How do STEM Undergraduates Structurally Conceive Change During Modeling?

Sindura Subanemy Kularajan
Texas State University

Jennifer A. Czocher
Texas State University

Using data from teaching experiments and theories from quantitative reasoning, we built second-order accounts of students' mathematics on how they conceived change via operating on existing quantities. We report three different ways STEM undergraduates structurally conceived change as they constructed mathematical models for real-world scenarios.

Keywords: Quantitative Reasoning, Mathematical Modeling, Change

A student's conceptualization of change plays a central role in their mathematical models they construct for real-world situations. Researchers have looked into how students' conceptualizations of change inform the mathematical decisions students make while modeling dynamic situations. For example, researchers have investigated students' conception of rate of change, limit, and accumulation through considering students' images of change in dynamic situations (e.g., Carlson et al., 2002; Thompson, 1994a). In addition, researchers have also studied how students' conceptions of change influence how students conceive variation and make sense of dynamic situations (Castillo-Garsow et al., 2013). However, these studies report on accounts of student thinking by focusing on students' variational and covariational reasoning as the student reasons about how values of a quantity are *changing*. We still need an account of how students conceive *change,* as a measurable attribute of a varying quantity, through establishing relations among existing quantities. Thus, we ask: how do STEM undergraduates conceive change while constructing mathematical models for real-world scenarios?

Theoretical Underpinnings

Our research lies within the cognitive perspective of mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling comprises the cognitive processes involved in constructing a mathematical model for real-world scenarios. We define a mathematical model to be a representation of the relations among conceived quantities.

Quantities are conceptual entities that exist in the mind of an individual. Thompson (1994b) defined *quantity* as a mental construct of a measurable attribute. It consists of three interdependent entities: an object, a measurable attribute, and a quantification. *Quantification* involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between the attribute's measure and the unit of measure (Thompson, 2011). Quantification takes place in the mind of an individual and, as observers, we can only use an individual's externalized actions to infer whether and how she has quantified an attribute. Czocher & Hardison (2021) presented eight observational criteria that can be taken as indication that a student has conceived measurement process for a measurable attribute of an object.

A relation among measurable attributes is established through operating on quantities. Thompson (1994b) defines *quantitative operation* as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a *quantitative relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994b). In other words, a quantitative relationship is the "conception of three quantities, two of which determine the third by a quantitative operation (Thompson, 1990, p. 12)." Examples of quantitative operations include *combining two quantities additively*, *comparing two quantities* *additively, combining two quantities multiplicatively*, *comparing two quantities multiplicatively*, *instantiating a rate*, *generalize a ratio*, and *composing two rates or ratios* (Thompson, 1994b). Thompson defined (1990) defined *quantitative structure* as a network of quantitative relationships. For example, the total number of feral cats that visited a backyard during one weekend is a quantity that may be constructed by *additively combining* the number of cats that visited the backyard on Saturday and the number of feral cats that visited the backyard on Sunday. At the same time, the total number of feral cats that visited the backyard on Saturday during the time period 9am to 5pm is a quantity that maybe constructed by *instantiating a rate* of 10 cat-bird interactions per hour for 8 hours.

Borrowing ideas from the aforementioned constructs we define *structural conception of change* as conceiving change as a measurable attribute of an object through forming a relation among constituent quantities (i.e., operating on quantities). We operationalize change as a measurable attribute of a varying quantity, that can be quantified via a quantitative relationship. The research question addressed in this report is: In what ways do STEM undergraduates structurally conceive change while constructing mathematical models for real-world scenarios?

Methods

We present data from a pair of 10-hour individual teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors at a large university. The overall goal of the teaching experiment was to investigate how students conceive real-world situations through quantities and relations among quantities. Our participants, Szeth and Pai were both enrolled in differential equations at the time of the interviews. Throughout the teaching experiment, Szeth and Pai worked on 10 and 9 tasks, respectively, that were based on real-world scenarios. In this report we present data from *The Disease Transmission Task*, *The Pruning Task*, and *The Population Dynamics Task*. We focus on these tasks because they exemplify how Pai and Szeth operated on quantities that they had previously constructed, to structurally conceive change.

The Disease Transmission Task: Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model that informs us about the dynamics of how the disease would spread through the population.

The Pruning Task: Imagine you have a hedge in your garden of some size, S, and you want it to increase its size even more. You hire a gardener for some advice on growing this particular plant. She advises you that the overall rate of growth will depend both on the extent of pruning and on the regrowth rate, which is particular to the plant species and environmental conditions. Both rates can be measured as a percentage of the size of the plant. The pruning rate can be adjusted to result in a target overall growth rate. Can you derive a model for the rate of change of the size of the plant?

The Population Dynamics Task: Suppose in a laboratory setting, we are looking at large populations of breeding stock in which individuals give birth to new offspring but also die after some time. Suppose that on average each member of the population gives birth to the same number of offspring, α, each season. The constant α is called per–capita birth rate. We also define β as the probability that an individual will die by natural causes before the next breeding season and call it the per-capita death rate. Construct a mathematical model that describes the dynamics of this population.

The interviews were retrospectively analyzed to construct second-order accounts (Steffe & Thompson, 2000) of students' reasonings via inferences made from students' observable activities such as verbal descriptions, language, written work, discourse, and gestures. The retrospective analysis consisted of multiple passes of through the data to arrive at examples that illustrate the different ways Szeth and Pai structurally conceived change. First, we watched the videos in MAXQDA in chronological order and paraphrased each interview by chunks. Next, we created accounts of students' mathematics and the reasons they attributed to their mathematics. Next, we went over the accounts we created and the videos at the same time and refined our accounts by adding details using theories from quantitative and covariational reasoning. We credited a student to have instantiated a quantity if we were able to infer from his reasonings that he had conceived an object, attribute, and a measurement process for the attribute. As evidence of student to have conceived a measurement process, we checked if at least one of the quantifications criteria was met (see Czocher & Hardison, 2021). We used segments of transcripts, where the students engaged in quantitative reasoning along with inscriptions and gestures as evidence for our claims. Next, from these accounts, we selected instances where the students engaged in structurally conceiving change. Next, while watching the videos and going over our accounts of students' structural conception of change, we further refined our accounts by adding details of how Pai and Szeth operated on constituent quantities to form a relation in order to conceive change. Next, we went through our accounts of Pai's and Szeth's structural conception of change and observed any patterns, in terms of operation on constituent quantities, that were consistent throughout the sessions. We made note of them by summarizing the pattern. We watched the videos in their entirety again and made sure all of the different ways of conceiving change were recorded. Finally, we refined our second-order accounts by seeking clarification on utterance and gestures to support our claims on Pai's and Szeth's structural conception of change. We present some of these second-order accounts below.

Findings

Pai structurally conceived change as accrual and as the additive comparison of quantities. In contrast, Szeth structurally conceived change as an instantiation of rate. I illustrate these structural conceptions using Pai's work from *The Disease Transmission Task* and *The Pruning Task* and Szeth's work from *The Population Dynamics Task*.

Change as the Accrual

Conceiving change as the accrual entails an image of accruing the amounts at different points in time, by additively combining the amount present before and the amount that was newly added. In *The Disease Transmission Task*, Pai constructed the expressions in Figure 1 to represent the change in healthy people and change in infected people.

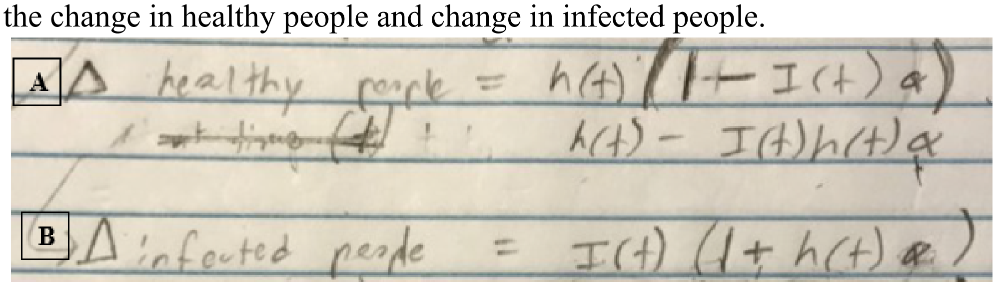

Figure 1. Pai's expression for the change in infected people and change in healthy people

After Pai constructed the above expressions, we asked him by how much the infected people would change. The goal of this question was to document how Pai was distinguishing between "$\Delta\, infected\ people$" and $I(t){\cdot}h(t){\cdot}\alpha$, because these two quantities carried the same meaning to us. Interestingly, Pai said that the "increase [in infected people]" was $I(t){\cdot}h(t){\cdot}\alpha$. Even though we used the term change when we asked him by how much the infected people changed, Pai replied with the "increase" in infected people. We interpret that Pai associated two different meanings to the attributes change and increase of the infected people, while he continued to associate change in infected people with expression B in Figure 1. This interpretation was further evidenced in the excerpt below.

Pai: Well, yeah. Change in the infected people is the base infected people at time t, plus those again, infected at time t. And this is just an increase. $h(t)\cdot I(t)\cdot \alpha$. Is going to be less than... You're going to be adding this to...to the base population.

Interviewer: Okay. I see. So yeah, you are seeing the delta infected people as the change in that sense, the total number of infected people that would be at the end of an infectious period?

Pai: Right.

Interviewer: Whereas the other one, how much... How many are getting infected that you have to add to the base.

Pai: Yeah, it's the increase, but just the increase of people infected in this time period.

Interviewer: Okay.

Pai: Maybe I could write that better saying this is the people infected in time period t [pointing at the expression $h(t)\cdot I(t)\cdot \alpha$]. That was a better statement of it. Like people infected... Well, people infected... In current period t. Whereas $I(t)$ is just how much we had already. Initially, before interactions with other healthy people. If that makes sense.

From the above excerpt, we interpret that Pai quantified change in infected people and increase in infected people differently. While Pai quantified the change in infected people as the "base population" plus how many more got infected, he quantified the increase in infected population as just the how many more people got infected, which to him was $h(t)\cdot I(t)\cdot \alpha$.

Pai continued to reason as to why expression B in Figure 1 represented the change in infected people as "cause that's a change in two - over two time periods because you're adding one. Cause you're adding another time period." From his reasoning, we interpret that Pai imagined to be representing the change in infected people as an accrual of the infected population at different points in time. For Pai, $I(t) + h(t)\cdot I(t)\cdot \alpha$ represented the change in infected population as an accrual of the amounts of infected people across a progression of time. On the other hand, for Pai, $h(t)\cdot I(t)\cdot \alpha$ represented increase in infected people during the moment of interest. Therefore, Pai structurally conceived change in infected people as the *additive combination* of the amount of infected people present at the beginning of the moment of interest and amount of people that got infected at the moment of interest.

Change as the Additive Comparison of Quantities

Structurally conceiving change as the additive comparison of quantities entails constructing change in the amount during a time period $[t_1, t_2]$ as the additive comparison of the amounts at t_1 and t_2, respectively. In *The Pruning Task*, Pai constructed the following expressions to represent the size of the plant at the end of the first and second pruning periods, respectively.

$$S_1 = S_0 - p(S_0) + r(S_0)$$
$$S_2 = S_1 - p(S_1) + r(S_1)$$

In the above expressions, Pai defined S_0 as the size of the plant at the beginning of the first pruning period, S_1 as the size of the plant at the end of the first pruning period, S_2 as the size of the plant at the end of the second pruning period, r as regrowth rate of the plant, and p as the extent of pruning. After constructing the above expressions, Pai constructed the expression below to be the change in size of the plant during the second pruning by *additively comparing* the quantities S_1 and S_2.

$$\Delta S = \ - p\left(S_1\right) + r\left(S_1\right)$$

Next, Pai constructed a general expression for the change in size of the plant during any pruning period of 1 unit, to be the following expression.

$$\Delta S = \ - p\left(S_t\right) + r\left(S_t\right) \qquad (1)$$

In the above expressions, Pai structurally conceived the change in size of the plant during a particular period by *additively comparing* the sizes of the plant at the beginning and end of the particular pruning period. Pai's structural conception of change, in this manner was also prevalent in the remaining of his work in this task. We illustrate this episode below.

Pai started working on deriving an expression for the rate at which the size of the plant was changing with respect to time. As Pai was not sure how to proceed, the interviewer directed him to derive an expression for the change of plant's size during a pruning period of length Δt. The intent of this question was to guide Pai towards thinking of the average rate of change of the size of the plant and how it can be manipulated to derive the instantaneous rate of change. Pai constructed the expression below to represent the change in size of the plant during two successive pruning periods.

$$\Delta S =- p\cdot\left(S_{t+1}\right) + r\cdot\left(S_{t+1}\right) - \left(- p\cdot\left(S_t\right) + r\cdot\left(S_t\right)\right) \qquad (2)$$

Although expression 2, for Pai, represented the change in size of the plant during two successive pruning periods, normatively it represents how much more the plant changed in size than the previous pruning period.

The interviewer emphasized to Pai that expression 1 is the change in size of the plant during any one pruning period and asked what the change in size of the plant would be during two such pruning periods, given that the change is the same for each pruning period. The intent of the question was to give Pai the opportunity to structurally conceive overall change during two successive pruning periods as the additive combination of the changes in size of the plant, during each pruning period. However, Pai was still referring expression 2 to as the change in size of the plant during two pruning periods. Pai gave the following reasoning:

> *Pai*: Because it's a change of the size of the plant over two periods. This is the change of... It's the second pruning period, $t + 1$ [*pointing at* $- p\left(S_{t+1}\right) + r\left(S_{t+1}\right)$], minus the initial pruning period change [*pointing at* $- p\left(S_t\right) + r\left(S_t\right)$].

When the interviewer pointed out that expression 2, to her, represented a "change minus change," Pai recognized expression 2 did not achieve his goal.

In an attempt to reduce Pai's cognitive load and to aid in unpacking his conception of change, the interviewer modified the scenario to be such that the change in size of the plant during a pruning period would be a constant β instead of ΔS. The interviewer asked Pai what the change in size of the plant would be over two time periods, given that β is the change during each pruning period. Pai said that it would be "The beta of the second pruning period minus beta of the first pruning period". This again serves as evidence that Pai's structurally conceived of change as the *additive comparison* of quantities.

The interviewer pointed out to the fact the quantity he constructed – "the beta of the second pruning period minus beta of the first pruning period" – was change in change. She asked what the overall change would be from the beginning of the first pruning period to the end of the second pruning period. Pai's responded with "You take the end, the change... Well, take the size at the end of period two minus the size of initial before period one. That's overall change." Again, Pai was only measuring the attribute change in size of the plant through comparing the sizes of the plant at different times.

To circumvent Pai's thinking of change as "beta of the second period minus beta of the first period" the interviewer shifted his focus to the quantities S_1, S_2 . The interviewer asked Pai that given $\beta = S_2 - S_1 = S_3 - S_2$, and so on, what the total change in size of the plant be from the beginning of the first pruning period to the end of the third period. To that, Pai answered $S_3 - S_1$ (Figure 2, point A). The fact that Pai was still considering the difference between S_1 and S_2 (instead of $\beta + \beta$) to find the change in size of the plant confirms our conjecture that his structural conception of change predominantly entails the *additive comparison* of the amounts at different points in time. When the interviewer asked him what $S_3 - S_1$ was in terms of β, Pai solved for the system of linear equations to get 2β. He attained 2β through comparing the quantities S_3 and S_1 and not by adding the $\beta's$ during each compounding period (See Figure 2). Only after arriving at 2β, Pai realized that it is "two changes in size β".

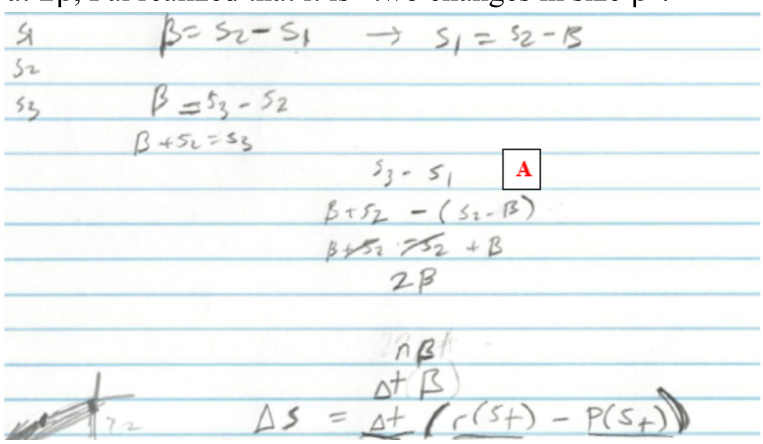

Figure 2. Pai additively compared the sizes of the plant at different times to evaluate the change in size of the plant

Change as the Instantiation of Rate

Structurally conceiving overall change as instantiation of change entails constructing overall change through *multiplicatively combining* the change during an interval of one unit of time and the length of the time period. In *The Population Dynamics Task*, Szeth constructed the expression in Figure 4 to represent the "size of the population at time t."

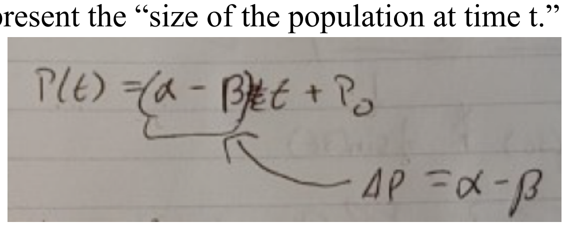

Figure 4. Szeth's expression for the size of the population at time t

In the expression in figure 4, Szeth defined $P(t)$ as the "size of the population at time t," P_0 as the population "that is already there" and α and β were given in the task as the per-capita birth rate of the population and the per-capita death rate of the population of the population, respectively. In the above expression in Figure 4, Szeth nominalized $(\alpha - \beta)\cdot t$ as "how much it [*the population*] is growing by," and $\Delta P = \alpha - \beta$ as the change in the population during one season. Szeth explained that he multiplied $\alpha - \beta$ by t because he wanted a "method within the equation for it to iterate over time [because $\alpha - \beta$] is based on the one season." When Szeth referred to $\alpha - \beta$ as "based on one season" and denoted $\Delta P = \alpha - \beta$, Szeth conceived the change in the size of the population during an interval of one unit of time as $\Delta P = \alpha - \beta$. Then to construct overall change in size of the population during a time period, Szeth *multiplicatively combined* the net change in the population during an interval of one unit of time and the length of the time duration. For Szeth, the length of the time duration was from $t_i = 0$ to $t_f = t$.

Discussion

We presented three examples of differing ways that Pai and Szeth structurally conceived change in terms of operating on constituent quantities. Structurally conceiving change as the accrual entails constructing an image of accruing the amounts at different points in time, by additively combining the amount present before and the amount that was newly generated. Structurally conceiving change as the additive comparison of quantities entails constructing change in the amount during a time period $[t_1,t_2]$ as the additive comparison of the amounts at t_1 and t_2. Finally, structurally conceiving overall change as instantiation of change entails constructing overall change through multiplicatively combining the change during an interval of one unit of time and the length of the time period.

In *The Disease Transmission Task*, Pai structurally conceived change in as an accrual through *additively combining* the amount of infected people at the beginning of the moment of interest and the amount of people who got infected at the moment of interest to construct the change in the amount of infected people. However, in *The Pruning Task*, despite the interviewer's efforts in having Pai to construct change in the size of the plant during two pruning period through additively combining the change in the size of plant during each pruning period, Pai consistently constructed change as an *additive comparison* of quantities. We conjecture that Pai had a strong preference for structurally conceiving change as the additive comparison of quantities, that he dropped the situational attributes of the quantities operated on and the resultant quantities were measuring. In contrast, Szeth structurally conceived change as an instantiation of rate where he unitized the change in size of the population and then multiplied the unit change by the length of the time period.

Knowing the different ways students structurally conceive change provides insights about the mathematical decisions students make during their mathematical modeling activities. This knowledge has implications for the teaching of mathematical modeling. For example, for a student who structurally conceives change as an instantiation of rate, like Szeth, the construction of instantaneous rate of change – typically the desired output while modeling dynamic situations – can be scaffolded by first constructing the change during a time period to the average rate of change during a time period, and finally to the instantaneous rate of change by considering infinitesimally small time periods. For someone who conceives change as an accrual, like Pai, focusing on the amount increased/decreased may yield favorable modeling outcomes. As quantities are building blocks of a mathematical model (Larson, 2013), understanding how students operate on existing quantities to construct a new quantity informs researchers and educators *where* and *how* to intervene in the students' mathematical modeling process.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1750813.

References

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31–37.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378.

Czocher, J. A., & Hardison, H. L. (2021). Attending to quantities through the modelling space. In F. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West*. Springer.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 267–291). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Larson, C. (2013). Modeling and quantitative reasoning: The summer jobs problem. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 111–118). Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum

Thompson, P. W. (1990). A theoretical model of quantity–based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University.

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics,* 26(2-3), 229-214.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 33–57). University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for research in mathematics education*, 421-456.