Students Leverage Struggles with Proof to Script Fictional Dialogues about the Rules of Proving

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Research has noted that some aspects of proof remain implicit in undergraduate instruction. Accordingly, we propose that direct interaction with the rules of proving (e.g., deductive reasoning, proof-writing conventions) is of didactical value to students transitioning to proof-based mathematics. Using the commognitive framework, we aim to delineate the rules that newcomers to proof find challenging. These were solicited with a scriptwriting task that asked students to create a fictional dialogue about a proof-related mistake that they had experienced. The task was embedded in a homework assignment in an undergraduate course for academically motivated high-schoolers. Thematic analysis of 61 scripts resulted in a range of rules that concerned the use of examples, logical circularity, and proof layout. We discuss these findings in relation to students' instruction and the existing literature on proof.

Keywords: Commognitive framework, proof, rules of proving, scriptwriting, transition to proof

Introduction

Dreyfus (1999), Selden and Selden (2015), and other researchers have argued that many aspects of proof remain implicit in undergraduate teaching and learning. Consistently, the lion's share of proof research can be reframed as students grappling with new mathematical rules that are not as explicit as they could be (e.g., Balacheff, 2008; Harel & Sowder, 2007; Moore, 1994; Selden & Selden, 1987; Weber, 2002). Overall, the shift to proof-based mathematics requires a fundamental change in the rules of argumentation and justification. Kjeldsen and Blomhøj (2011) maintain that developing appropriate rules for activities of this sort is indispensable for mathematics learning. In the context of proof, such rules can be associated with deductive reasoning, propositional calculus, logic, and generally accepted proof-writing conventions. Accordingly, we propose that a direct engagement with the *rules of proving* (*RoP* hereafter) is of didactical value to newcomers to proof-based mathematics. This study comes from a larger project that explores how newcomers articulate, apply, and make sense of RoP while making their first proving steps.

Background

A "solid finding" of mathematics education research is that "many students rely on validation by means of one or several examples to support general statements, [and] this phenomenon is persistent in the sense that many students continue to do so even after explicit instruction about the nature of mathematical proof" (ECEMS, 2011, p. 50). Another occasionally reported issue pertains to logical circularity. In abstract algebra, Selden and Selden (1987) found that some students use one version of the conclusion of the assigned statement to prove an equivalent version of the same conclusion. Additional findings of a similar type have been reported in the literature (e.g., Weber, 2002).

Emerging from cognitive studies, these findings are often framed in terms of students' errors and misconceptions since they have to do with proof validity. We adhere to the social perspective on proof (Stylianides et al., 2017). Within this perspective, the findings can be viewed as students' deviations from the rules that are broadly accepted in professional mathematics communities. And while it is not rare for mathematicians to disagree on proof-related issues (e.g., Lew & Mejía-Ramos, 2020), abiding by such rules as modus ponens and propositional calculus is likely to be expected in all communities that practise proof.

Generating proofs that abide by these rules requires additional rules that are less universal. For instance, Stylianides (2007) argues that in a classroom, a proof uses accepted statements and employs forms of reasoning and expression that are known to, or are within, the conceptual reach of the students. Accordingly, proofs can vary depending on what each classroom renders "accepted," "known," and "within students' reach." Furthermore, guidelines are needed to compile mathematical statements, forms of reasoning and expressions into a proof that the classroom community will endorse. In terms of Yackel and Cobb (1996), RoP of this kind are not very different from sociomathematical norms that determine "what counts as an acceptable mathematical explanation and justification" (p. 461). While using different terms, Dreyfus (1999) discusses instances where students grapple with such community-specific rules in the context of Calculus and Linear Algebra. The instances contain explanations and proofs that Dreyfus describes as insufficient: they present computations without accompanying explanations, draw on theorems without spelling them out, and, overall, do not "go back" enough and do not "go deep" enough.

The findings overviewed in this section come from research where students did not follow RoP that the researchers viewed as conventional. This study endorses an anti-deficit approach that foregrounds productive resources that all students bring to mathematics learning. The approach acknowledges that "learning takes time and imperfect articulations of mathematical ideas and some inconsistencies in the student's current conception are a natural part of the process of learning" (Adiredja, 2019, pp. 416–417). Specifically, we interpret students' deviations from conventional rules as a marker of their ongoing transition to proof-based mathematics. We also see value in encouraging students to notice and reflect on such deviations as a stepping-stone towards the conventional RoP (e.g., CUPS, 2004). This study aims *to delineate RoP that newcomers to proof-based mathematics find challenging*.

Commognitive Framework

We follow in the footsteps of previous studies that mobilized the conceptual and analytical potential of commognition to study university students' engagement with proof (e.g., Brown, 2018 Kontorovich, 2021, Kontorovich & Greenwood, 2022; Kontorovich et al., 2023). Commognition construes mathematics as a *discourse*. Discourses are defined as, "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard 2008, p. 93). Then, *learning* is a process through which one's communication starts abiding by the rules of the target discourse.

Proof and proving concern *endorsable narratives*—narratives that can be rendered as true "according to well-defined rules of the given mathematical discourse" (Sfard, 2008, p. 224). Sfard (2008) acknowledges that "being dependent on what participants find convincing, routines of substantiation are probably the least uniform aspect of mathematical discourses. The very term endorsement may be interpreted differently by different people" (pp. 231–232). Indeed, the mathematics community has been revising these interpretations of what constitutes endorsement throughout history (e.g., Kleiner, 1991). Notwithstanding, Sfard (2008) proposes that "for today's mathematicians, the only admissible type of substantiation consists in manipulation on narratives, and it is thus purely intradiscursive" (p. 232). This manipulation unfolds as a sequence of utterances (or proof components), each either an "accepted fact," or derived according to some generally accepted rules (e.g., modus ponens). In this way, we associate RoP with intradiscursive principles for endorsing true mathematical statements and rejecting false ones. Colloquially speaking, these rules pertain to “when to do what and how to do it” (Bauersfeld, 1993, p. 4).

Commognition distinguishes between *object-level rules* and *meta-discursive rules* (or *metarules* for short). The former rules attest to the properties of mathematical objects and take the form of narratives on these objects (e.g., “the product of rational numbers is rational”). Metarules focus on how discourse participants formulate and substantiate object-level rules. Thus, RoP are meta-discursive by definition. Metarules are often enacted tacitly. Once a rule has been captured in words (i.e. when a *rule-narrative* is generated), commognition refers to it as *endorsed* by the interlocutor (Sfard, 2008). In this study, we scrutinize RoP that newcomers to the discourse of proof-based mathematics endorse.

The Study

Our participants were students in a special program entitled “Mathematics extension and acceleration” in a large New Zealand university. The program is intended for mathematically motivated and academically inclined students (usually 17-year-olds) in their final year of high school. As part of this program, the students take a special semester-long mathematics course that gives them academic credit for a bachelor’s degree in mathematics or engineering.

Proofs play a marginal role in New Zealand school mathematics (Knox & Kontorovich, 2022). The first part of the course ushers students into proving, while leveraging their familiarity with different kinds of numbers. In the first two lectures of the course that we studied, the instructor introduced proof as “an argument that mathematicians use to show that something is true.” Then he presented three true mathematical statements and led a whole-class discussion about what he positioned as “broken proofs.” In the following three lectures, he introduced selected properties of real numbers (e.g., distributivity) and illustrated how these could be used to derive additional properties (e.g., $-a = -1 \cdot a$); the illustrations were discussed as instances of “good proofs.” In these lessons, the instructor shared many proof-related insights and guidelines, including: examples do not prove universal statements, avoid circularity, start with true things, delineate the statement to be proved before proving it, the scope of the introduced variables should be defined, use words to explain the proof (not just formulas), avoid proofs that prove without explaining. In the last section, we draw on these insights and guidelines to reflect on RoP that students endorsed.

A *scriptwriting task* was designed to pursue the aim of this study. In tasks of this sort, students are provided with a thought-provoking prompt and asked to offer a resolution by scripting a fictional dialogue between made-up characters (Zazkis et al., 2013). Research has used scriptwriting to investigate students’ proving (e.g., Gholamazad, 2007). Brown (2018) argued that by making students’ envisioned interactions about a proof public, scriptwriting affords students a way to make their thinking visible and fosters reflection. In turn, researchers get access to proof-related aspects that students render important.

The course instructor and the first author collaboratively developed a scriptwriting task to provide students with opportunities to engage with RoP and collect data (Kontorovich & Bartlett, 2021). Figure 1 presents the task eventually developed. It was embedded in an individual homework assignment together with proof-requiring problems that are typical to transition-to-proof courses.

The second author conducted an initial analysis of students’ submissions (Liu, 2018). The analysis started with an overview of the 74 scripts that were collected to develop a general impression of whether they pertain to RoP. This process converged into 61 submissions, which underwent a thematic analysis (e.g., Braun & Clarke, 2006). The initial themes were iteratively separated and merged to develop general categories and descriptions of rule-narratives that the students generated through the voices of characters that represented them in their scripts (*Student-characters*, hereafter). This process combined deductive and inductive techniques as the initial themes were informed by the literature as well as by the content of students' scripts. When characterizing students' rule-narratives, we distinguished between the *restricting* and the *guiding* version of a rule: the former posited that something should not be done, while the latter offered guidance for how proving should be pursued.

> Writing proofs can be quite difficult, especially when you're new to them! One of the best ways to get better at writing proofs is to **look back** at past mistakes and think about how to fix them. This exercise is meant to help you do this, as follows:
> (a) Come up with a mistake you've made (possibly while working on this assignment!) or seen others make when trying to prove something.
> (b) Write a dialogue between you and an (imaginary) friend, where they make this mistake and you explain what's wrong with that idea and how to fix it: i.e. something like
>
> *Paddy's imaginary friend*: So, I was trying that checkerboard problem above[4], and I figured that if I wrote that I tried three different ways to do it and none of them worked that would totally count as a proof.
>
> *Paddy*: Well, actually it probably doesn't. You see . . .

Figure 1. A scriptwriting task[1]

Findings

This section presents six categories of the most common rules that emerged from analyzing students' scripts. We use rule-like formulations for the first five kinds due to the high consistency of the rule-narratives that the students' generated. Next, we elaborate on each kind and illustrate it with representative examples from students' scripts.

Examples are insufficient to prove a universal statement

This type of rule featured in 16 scripts. There, the students' rule-narratives stressed the insufficiency of drawing on specific examples to prove a universal statement that pertains to infinitely many cases. For example, one script revolved around the statement that the product of two rational numbers is rational. To prove it, "an imaginary friend" multiplied the rational $\frac{3}{4}$ and $\frac{4}{2}$ to obtain $\frac{12}{4}$, which he identified as a rational number. The Student-character did not disagree with the rationality of the product, but he did not accept that this substantiation can be endorsed as a proof. Specifically, the Student-character generated a rule-narrative, stating that "Just because you gave an example where the claim is true, doesn't mean it's always true". In other scripts, the Student-characters explicated the insufficiency of supporting examples by stressing that "a proof cannot be made of citing examples only". Some rule-narratives highlighted that even a large number of supporting examples would not make a proof (e.g., "It doesn't matter how many examples you give […] that would still not be satisfactory enough to say that the claim is true"). Guiding rule-narratives featured in 5 scripts and suggested considering "the full range" and "all possible values" that are relevant for the focal statement.

[1] The task refers to a famous tiling chessboard problem that was part of students' homework assignment.

Failures to produce an example are insufficient to prove a non-existence statement

Six scripts highlighted the insufficiency of failing attempts to exemplify a non-existence statement to be considered as a proof. Consider an excerpt:

Friend- character: When I was attempting the chessboard problem, I tried to manually place rectangles onto the box and fit them in. Since I couldn't solve it, it proves the problem is unsolvable.

Student-character: When you tried to place rectangles on the board, it only proves the specific arrangement doesn't work. But there are many different arrangements which can be attempted, and the result of one cannot be assumed to be the same as all the others.

In this exchange, we see the Student-character generating a rule-narrative, positing that the friend's unsuccess in covering the board does not account for all possible arrangements. Then, her attempts cannot be viewed as a valid proof that the board cannot be covered. In the first part of her submission, the same student wrote this explicitly, "[the specific arrangements] showed that they could not fit, but it did not prove the problem is unsolvable". Another Student-character argued that "no matter how many examples you provide, it's possible for a solution to exist". In this way, other Student-characters also stressed that specific attempts to generate examples for a non-existence statement does not extrapolate to other possibilities. Contradiction-based guiding rules-narratives appeared in three scripts in this category.

Deriving a true claim from a statement does not prove it

This type of rule emerged from seven scripts where "imaginary friends" manipulated the assigned statement until the obtainment of a true claim. Then, it was interpreted as a warrant that the initial statement has been proved. For instance, one "imaginary friend" drew on her deriving that "$0 = 0$" to argue that some initial (and never explicated statement) was true. But the Student-character insisted that "Proofs don't work like that […and] you definitely cannot say $0 = 0$ so it must be true". In what appears as an attempt to replicate this approach, the Student-character wrote "1=2" and multiplied both sides by zero to end up with "$0 = 0$". This move illustrated that a deviation from the focal rule opened the door to the endorsement of false statements. Similar rule-narratives featured in other scripts, arguing that "Assuming the truth of the claim and manipulating it to reach a true conclusion does not prove that it is true." Instead, four scripts contained guiding rules, suggesting one should "start with what you know is truth."

A proof must contain no invalid moves nor false claims

This type of rule grew from 12 scripts where "imaginary friends" conducted a mathematically invalid move to prove a statement. These moves included multiplying both parts of the equation by zero, manipulating with infinity and 0^{-1} like these were real numbers, and building on false assumptions that steered proving towards a problematic course.

To diversify the scope of illustrations, we refer to a script where this rule was enacted silently without being endorsed. There, the 'Imaginary friend' proposed to prove the irrationality of $2^{\frac{1}{\sqrt{2}}}$ by first showing that the exponent is irrational, and then proving that a rational number raised to an irrational power is irrational. The Student-character rejected "that method" by counter-exemplifying the second step. In this way, the scriptwriter implicitly suggested that building on a false assumption disqualifies the proof. Other students were more explicit regarding this rule. In one script, the Student-character posited that "Unfortunately, we can't divide by 0, which kills the whole argument". In another submission, a student shared "I made a false assumption that made the proof false", which tacitly foregrounds the focal rule. A guiding rule-narrative featured in a single script, and it suggested "always start with something that you know is true!"

A proof should not include circularity

This type of rules emerged from six scripts where an "imaginary friend" drew on the assigned mathematical statement or some variation of it within the proof itself. Then, the Student-character called out this circularity, arguing that it disqualified the proof. For instance, in one script, a "friend" attempted to prove that the sum of irrational numbers is irrational. As part of it, they defined the irrational x and y, and sequentially relied on the irrationality of $x + y + (-x)$ "as claimed" in the original statement. In turn, the Student-character responded with a rule-narrative stating that "you cannot make assumptions based on the claim [since you] are proving the claim to be true". Additional examples of rule-narratives that the students generated included "using what is required to be shown/proven in the proof (!), which is clearly invalid" and "Well you are trying to prove that the product of two rational numbers is also rational. So how can you say that $a \cdot m$ is rational, since this is the fact you are trying to prove [a and m were previously assumed to be rational]?" Only in one script, a Student-character proposed a guiding rule, calling their fictional peer to "stick to things that you 100% know are true: like our axioms".

The rules of proof layout

This category emerged from 18 scripts. The rule-narratives that the students generated in these scripts were diverse, and we sub-divided them into four types: (i) the statement to be proved should be explicated, (ii) the numeric sets of the introduced variables should be specified, (iii) the proof moves need to be explained, and (iv) sub-narratives constituting a proof should be proved. Table 1 illustrates each rule with excepts from students' scripts. Overall, these rules regard the layout of different proof components. Note that all these rules are guiding.

Table 1. Rules of proof layout

Sub-categories of the rules	Excerpts from students' scripts
(i) The statement to be proved should be explicated	"You must have a claim at the top of your proof so that other readers and markers can understand what you are trying to do. Plus your conclusion may not make much sense without any claim to measure it against"
(ii) The numeric sets of the introduced variables should be specified	"I forgot that when saying something is rational, you say it $= \frac{m}{n}$ where $m, n \in \mathbb{Z}$ you say $n \neq 0$"
(iii) The proof steps need to be explained	"You need to write down your logic and reason step-by-step to backup your proof."
(iv) Sub-components of a proof should be proved	"We cannot just assume that $\sqrt{8}$ is irrational. You must prove it first, otherwise the rest of the proof is invalid."

We notice the categorical tone of the rule-narratives in this category, implying that the rules are unambiguous and can be enacted straightforwardly. This appears to be the case for (i) and (ii). Regarding (iii), in some scripts, Student-characters insisted on introducing "math terms" and "explanations" to symbolic substantiations. This afforded students to maintain that a proof that "doesn't use words" is problematic, without discussing how "wordy" a proof should be.

The issue of ambiguity is also relevant to rule (iv): what mathematical claims can be used without any substantiation, and what claims need to be proved separately? Indeed, one script revolved around the rejection of "the sum of rational numbers is rational." There, the Student-character criticized a counter-example where $1-\sqrt{2}$ was treated as an irrational number without explicit proof. Then, the same character suggested using $-\sqrt{2}$ instead, arguing that "it's still an irrational number and $\sqrt{2}+-\sqrt{2}=0$". What makes this script interesting is that in the course, neither the irrationality of $-\sqrt{2}$ nor the rationality of 0 were proven, but the student still endorsed these numbers as such.

Summary and Discussion

Building on the literature and the commognitive framework, we argued that direct engagement with RoP is of didactical value to students in transition to proof-based mathematics. To trigger this engagement, we asked students to script a fictional dialogue about a proof-related mistake that they experienced. We used the students' scripts to explore the metarules that newcomers to proof can find challenging.

The analysis resulted in six kinds of rules, none of which are foreign to the mathematics education literature. The innovation is in their epistemological status: while in most studies, the rules emerged from students' infelicitous proof attempts (e.g., Selden & Selden, 1987; Weber, 2002), in our case, the students were the ones to position these rules as something that a mathematical proof must avoid or abide by. Many students went beyond the rule formulation to illustrate, elaborate, and substantiate these rules. Accordingly, a key contribution of our study is in showing that newcomers can not only struggle with proof, but also communicate about it on a meta-level after a relatively short acquaintance period. That said, we acknowledge that our findings come from a special cohort that is hardly representative of a broader student population. Thus, we call for further anti-deficit research into how students learn RoP.

Sceptics may suggest that we should not "read too much" into students' endorsed rules as they could "simply" be drawing on what the course instructor demonstrated. Indeed, the overlap between the instructor's and students' rules cannot be ignored. However, even if the sceptics are correct and this is how students' rules came about, we argue that it is legitimate and even expected. Indeed, Sfard (2008) maintains that transitioning to a new discourse is impossible without a ritual phase, where learners imitate the discourse oldtimers. To do so, the students still needed to craft relevant proof attempts, generate their versions of rule-narratives, and apply them in a different context. Currently, we are developing an analytical framework to distinguish between cases of rigid imitation and agentive application of RoP. Overall, the identified overlap draws attention to the impact of proof instruction on students' communication about proof.

Each RoP can be captured in a guiding or in a restricting form. However, most of our participants adhered to the latter rather than the former. This result may be a side-effect of the assigned task: asking about a mistake may have nudged students to emphasize what is not supposed to happen in a proof. Alternatively, many guiding rules are based on deductive reasoning, communication about which requires a special vocabulary that the students may not yet possess. Accordingly, it could be interesting to explore whether students' capabilities to generate guiding and restricting rules develop in parallel.

We must be cautious when offering implications based on this exploratory study. Yet, we hope that its findings will encourage instructors to consider locating RoP at the center of their proof teaching. Furthermore, scriptwriting tasks may offer valuable information about students' emerging RoP: a carefully designed task can mirror the rules that students find challenging or ambigous, and can also point at the rules that currently escape students' attention.

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