Star Polygons: Symbolizing as Advancing Mathematical Activity in a Scripting Task

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Research indicates that investigating phenomena, rather than reproducing facts, should be a core experience in mathematics education. But despite its centrality to quality teacher education, it is still unclear how the effects of investigation tasks on teachers' mathematical development can be analyzed. In this study, we demonstrate the potential of scripting tasks as an avenue for capturing mathematical activity during an investigation. Participants, comprised of prospective and practicing teachers, explored the concept of a "star polygon" and recorded their deliberations, ideas, and conclusions as a scripted dialogue. We analyzed their submissions using the construct of advancing mathematical activity (Rasmussen et al., 2005; Rasmussen et al., 2015), focusing on how participants' interpretation and creation of symbols intersects other types of activity.

Keywords: Star Polygons, Scripting Tasks, Teacher Education, Advancing Mathematical Activity, Symbolizing

Introduction

Mathematics education research strongly advocates for learning environments centered on investigation, as opposed to techniques and procedures (e.g., Da Ponte, 2007, Yerushalmy, 2009; Leikin, 2014). Investigation tasks—also described as open problems or problem fields (Pehkomen, 2019)—require students to consider a particular object or a set of objects, collect data about them, identify their properties, and justify the discovered properties or relations. As such, conjecturing and then refining, proving, or refuting conjectures are the heart of investigation tasks.

There are multiple advantages for students engaged in mathematical investigations (e.g., Quinnell, 2010). For example, they develop a strong conceptual understanding, an awareness of the nature of mathematics as a discipline, and the ability to think creatively about mathematical ideas. Investigation-based learning environments also foster a supportive classroom atmosphere while simultaneously recreating the actual practice of mathematicians.

Numerous professional development efforts intend to prepare teachers for including problem solving and investigation tasks in their classroom and support them in this endeavor. However, for teachers to guide students' investigations, it is essential that teachers have experience in engaging in such activities themselves (e.g., Perdomo-Díaz et al, 2019). While mathematics educators agree on the importance of engaging teachers in mathematical investigations, there is no explicit guidance on how teachers' engagement in investigations can be studied.

Existing reports on teachers engaged in mathematical investigations employ close observations of teachers' work in class, video-recording of their work on a particular mathematical problem (e.g., Rott et al, 2021), and problem solving journals (Liljedahl, 2007) or portfolios (Gourdeau, 2019) in which research participants recorded their progress. In this study we present an additional approach—script-writing—for recording and analyzing teachers' engagement in a mathematical investigation. In doing so, we address the following research questions: *What aspects of participants' symbolizing activity are captured by scripted dialogues recreating their progress on an investigation task?*

Advancing Mathematical Activity

To answer this question, we turn to Rasmussen et al.'s (2005) alternative formulation of Tall's (1992) concept of advanced mathematical thinking that they call *advancing mathematical activity*. By focusing on mathematics that is "advancing," this construct allows researchers to study student activity that improves their understanding of mathematics, even if their work would not be considered "advanced" by some standard; by considering "activity" rather than "thinking," this construct attempts to capture a more holistic and diverse type of engagement with mathematics. Throughout their work, Rasmussen and colleagues identify four types of advancing mathematical activity that are socially or culturally situated practices: symbolizing, algorithmatizing, defining, and theoremizing (2005; 2015). In this report we focus on symbolizing.

Symbolizing "arises in part as a means to record reasoning" (Rasmussen et al., 2005; p. 57); that is, a symbol is a "notational device that [is] consistent with [a student's] mathematical reasoning," and symbolizing is the creation thereof (p. 58). The authors stress that they do not interpret symbolizing as necessarily an act of dissociation from context; designating a symbol as the representation of a quantity is not only a means of abstraction but also allows students to create, take ownership of, and share mathematical objects that characterize their thinking. Symbols, once refined, can act themselves as the input for future symbolizing activity. That is, symbols can manifest as a synthesis of multiple previous symbols.

Although we focus primarily on the presentation of symbolizing activity throughout this report, it is impossible to completely disentangle the types of advancing mathematical activity; with this in mind, we introduce the other types of activity as well. *Algorithmatizing* is the creation and discovery of (either local or global) algorithms. When engaged in *defining*, a student might construct a definition in light of examples and counterexamples; alternatively, they might leverage an existing definition in order to explore a related set of objects. Finally, theoremizing encompasses "both conjecturing and steps toward justifying the assertions" (Rasmussen et al., 2015; p. 264).

Methods

Participants and Setting

A total of 27 respondents participated in the study and, working in groups, produced 14 unique submissions; we refer to these as S-1 through S-14. The first seven submissions were generated by prospective teachers during a course on mathematical problem solving. This course, taken during the last term of the teacher certification program, leveraged problem solving activities to connect a variety of secondary and advanced topics in mathematics. The latter seven submissions were generated by practicing teachers who held at least a bachelor's degree in mathematics or science. The practicing teachers were enrolled in a professional development course that provided an overview of important topics and ideas in mathematics through the lens of pedagogy.

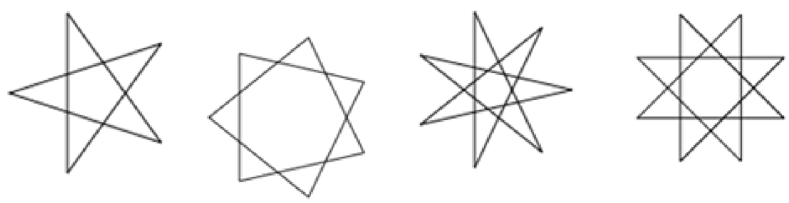
Participants' scripts often featured themselves as characters. To preserve their anonymity, throughout this report, named characters that appear in excerpts from submitted scripts have been given pseudonyms.

The Star Polygons Task

As part of their respective courses, both prospective and practicing teachers regularly engaged with *scripting tasks*; in such a task, participants are asked to write a script of a dialogue within a mathematical context. A scripting task might be used for lesson planning (where it is referred to as a *lesson play*, for example in Zazkis et al., 2009; Zazkis, et al., 2013). At other times, a scripting task is an activity that can be used to study how teachers respond to a student error (e.g., Zazkis et al., 2013), a student question (e.g., Bergman et al., in press; Marmur & Zazkis, 2018; Kontorovich & Zazkis, 2016), or an argument among the student characters (e.g., Marmur et al., 2020; Zazkis & Zazkis, 2014).

The *star polygons task* used in this study is a novel type of scripting task in that it prompted participants to record, as a script, their journey in a mathematical investigation. Preliminary pilot testing of a more open-ended version of the star polygons task showed that a certain amount of guidance was needed to support participants' initial engagement with the task. The revised task deployed in this study (Figure 1) provided a symbol system for naming and describing star polygons before asking participants to make sense of it. As we exhibit in the findings, this allowance facilitated communication and conjecturing without supplanting meaningful symbolizing activity.

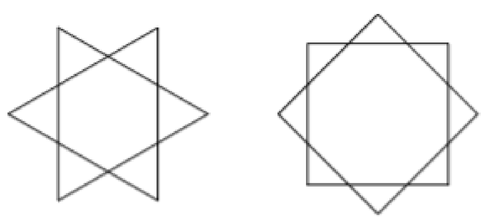
The figures below exemplify regular star polygons on 5, 7 and 8 vertices.



These are denoted – from left to right – as (5, 2), (7, 2), (7, 3) and (8,3). Explain the notation.

In (5,2) we refer to "2" as the number of "skips" or "steps" (skip to the second).

But these are NOT star polygons. Explain why.



Is there a star polygon on 10 vertices? On 12? On 30? On 31?

Explore different numbers. Describe your explorations and conclusions as a play, that is, in a form of a dialogue among problem solvers. Record your (your characters') conjectures and the ways of testing them.

Figure 1. The first part of the star polygons task.

The first part of the task introduced in Figure 1 engaged participants with investigating why select examples are not considered star polygons. Having justified this distinction, participants were instructed to explore whether a star polygon can be constructed on a given number of vertices.

A second part of the task, not pictured in Figure 1, prompted participants to describe a method for deciding if there exists a star polygon on N vertices and, if so, how many different star polygons on N vertices there are. Finally, participants exemplified their described strategy on a large number of vertices. Each part of the task was recorded as part of the scripted dialogues that comprised the collected submissions.

Data analysis

We initially analyzed participants' submissions using reflexive thematic analysis (Clarke & Braun, 2006, 2019; Nowell et al., 2017). That is, we hoped to allow the submissions themselves to guide our understanding of participants' mathematical activity as captured in their investigations. We began this process of immersion and interpretation by first reading and rereading the submitted scripts in order to thoroughly familiarize ourselves with the data. While doing so, we recognized that the most compelling aspect of the scripted dialogues was the description of the interaction between characters as they discussed and constructed a collective, social understanding and monitored each other's claims.

To emphasize the fundamentally communal nature of the activity reported by participants in their submissions, we leveraged Rasmussen and colleagues' (2005, 2015) four types of advancing mathematical activity. This decision was given credibility by the authors' emphasis that these are "acts of participation in a variety of different socially or culturally situated mathematical practices" (Rasmussen et al., 2005, p. 52). Furthermore, advancing mathematical activity has already been used to describe collective mathematical progress as part of an emergent framework (Rasmussen et al., 2015). We thus coded the data explicitly for instances of symbolizing, algorithmatizing, defining, and theoremizing. This process was guided by the definitions and examples of each practice presented in the advancing mathematical activity framework (Rasmussen et al., 2005; Rasmussen et al., 2015).

In the following section, we present instances of symbolizing activity that emerged during explorations of the star polygons task. Where appropriate, we point out where the symbolizing activity overlaps with other types of advancing mathematical activity. Finally, we pay particular attention to the ways in which participants' symbolizing activity was affected by their adoption of the prescribed (n, k) (n, k) notation within the task description.

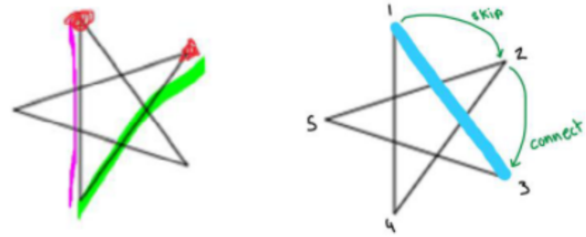
Findings

Evidence from student submissions suggests that the (n, k) (n, k) notation provided in the task description, despite providing a preconstructed symbol system to the participants, did not supplant participants' symbolizing activity. In fact, many submissions exhibited more robust symbolizing activity in direct response to the provided notation. For example, this occurred when participants used symbols to explain their intuitions about the notation to their groupmates. S-5 included the following exchange:

Martina: For the $(5,2)$ $(5,2)$ and $(7,2)$ $(7,2)$ stars, if you follow along and "draw" the lines, you are skipping 1 vertex to draw the next one. Maybe the 2 2 notates the number of steps you take as you draw the star polygon?

Bridget: Hmm let me know if I interpreted this the wrong way but here is what I got. To go from vertex Circle to vertex Triangle, I took the pink route then the green, so I counted those as two steps.

Joyce: Hmm, I was thinking about it differently. Here is a diagram of what I thought.



(a) Bridget's diagram (b) Joyce's diagram

Figure 2. Symbolizing activity in S-3.

Martina, Bridget, and Joyce demonstrate that there are multiple possible explanations for what constitutes a "skip" and use symbolizing activity to capture the underlying reasoning of each interpretation. Their discussion illustrates that participants still engaged in symbolizing activity in order to encapsulate and then compare their understanding of the (n,k) (n,k) notation within the context of the provided star polygon diagrams.

Other groups used symbolizing to hypothesize or reject alternative explanations for the provided notation. In S-13, a character proposes another interpretation of the k k in the (n,k) (n,k) symbol:

> *Abeni*: I thought the 5 5 represented the number of vertices and the 2 2 represented how many times there are 5 5 vertices. So this one being $(5,2)$ $(5,2)$ means that it has 2 2 sets of 5 5 vertices. See in the picture below using the different colours. There are 5 5 blue vertices and 5 5 red vertices. Two different sets of vertices all together.

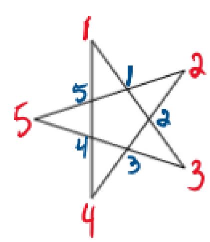


Figure 3. Abeni's suggestion for interpreting the (n,k) (n,k) notation in S-13.

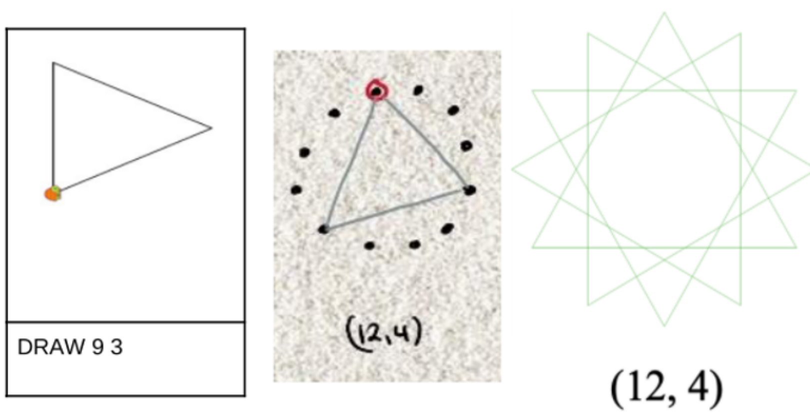
Abeni and her partner go on to draw and label multiple examples of star polygons, eventually concluding that both interpretations of the (n,k) (n,k) notation are valid. It is not until considering the provided non-examples that the characters recognize that Abeni's suggested interpretation cannot be used to differentiate these objects from the earlier examples. As Abeni observes, "My observation of 2 sets of 6 vertices holds true for this [the first non-example in Fig. 1]. So, I am not sure why it would not be a star polygon." In exploring this perceived contradiction, the characters of S-13 demonstrate how symbolizing can contribute to the act of defining.

Providing a symbolic notation had another unexpected benefit with respect to defining and theoremizing: many groups used the (n,k) (n,k) symbol as input for further symbolizing activity that facilitated the other types of advancing mathematical activity. For example, consider the following excerpt from S-5:

Matt: I think that we can have two overlapping star polygons that generate irregular star polygons if we can find a number (x, y) (x, y) which satisfies the equation $(x, y) = 2 \cdot (a, b)$ $(x, y) = 2 \cdot (a, b)$ where (a, b) is a star polygon. So, for example $(14,6) = 2 \cdot (7,3)$ $(14,6) = 2 \cdot (7,3)$, and (7,3) is a star polygon. So, (14,6) should look like 2 2 star polygons layered on top of each other.

Here, Matt manipulates and generalizes the (n, k) (n, k) symbol through further symbolizing activity in order to apply that notation to a newly defined category of non-examples ("irregular star polygons"). After defining this object, Matt generates a conjecture about particular combinations of n n and k k and applies their symbol system to specific examples. In this way, Matt's symbolizing allowed him to better engage with defining and theoremizing activity.

Perhaps the most interesting symbolizing activity occurred when participants interpreted the (n, k) (n, k) symbol as instructions for executing an algorithm for constructing additional star polygons. The symbols that they generated as a result of this algorithm were simultaneously products of their algorithmatizing activity and symbols used to explain the rationale for their algorithms. Consider the following three symbols produced by S-14, S-9, and S-2 provided in Figure 4.



(a) S-14's diagram (b) S-9's diagram (c) S-2's diagram

Figure 4. Various applications of the (n, k) (n, k) *symbol as an algorithm.*

The characters in S-14 and S-2 used two different computer software systems to generate their symbols (Figure 4(a) and 4(c), respectively), whereas characters in S-9 produced their symbol (Figure 4(b)) by hand. Each final symbol is a composition of an essential symbolic component (the equilateral triangle) and various secondary symbolic elements (e.g., "unused" vertices, additional overlapping triangles, indications of a "starting vertex"). In total, each symbol in Figure 4 encapsulates its respective group's understanding of how the notational symbol (n, k) (n, k) can be interpreted and applied as an algorithm. The differences in their interpretations, manifested as the secondary symbolic elements, affected how each group would go on to define when two star polygons should be fundamentally "different."

Discussion

Overall, our findings illustrate a variety of ways in which symbolizing activity manifested while participants investigated the star polygons task. Participants' efforts to interpret the provided (n, k) (n, k) notation facilitated their future exploration of non-examples of star polygons; in this way, symbolizing led to more robust defining. Participants also created new definitions when they manipulated the (n, k) (n, k) notation and used these observations to conjecture relationships between different star polygons. These conjectures and their justification link symbolizing to theoremizing. Finally, most participants treated the (n, k) (n, k) notation as an algorithm that described the construction of a star polygon. Clearly, when this was the case, symbolizing and algorithmatizing were intimately connected.

These observations, in sum, contribute towards answering our research question: *What aspects of participants' symbolizing activity are captured by scripted dialogues recreating their progress on an investigation task?* That is, scripting tasks effectively capture symbolizing activity in a variety of modalities. Furthermore, these different manifestations of symbolizing connected to and supported the other types of advancing mathematical activity. Existing research often acknowledges this interconnection between advancing mathematical activities; this study contributes to mathematics education by highlighting specific instances of these relationships.

The Effect of Technology

In our findings, we observed that the provision of a symbolic notation for naming and describing star polygons did not detract from participants' advancing mathematical activity in general, or their symbolizing in particular; in fact, it often contributed to it. Interpreting, manipulating, and extending this notation was a productive focus of the submissions.

Just as the introduction of this notation had a positive effect on participants' engagement with the task, some kind of technology for generating star polygons was equally invaluable—it allowed participants to generate examples on a large number of vertices without constructing them tediously (and sometimes incorrectly) by hand. However, we also draw attention to the ways that technology removed opportunities to engage in advancing mathematical activity.

For example, Figure 4(c) is a star polygon that was generated by an online applet. We note that the default functionality of this applet is to draw "compound" star polygons when the number of vertices shares a common factor with the number of skips, rather than to leave some vertices "empty" (cf. Figure 4(b)). By supplanting participants' symbolizing with this particular representation of the notation, technology had a perceptible effect on participants' later mathematical activities. For instance, the symbol $(10,4)$ $(10,4)$ was interpreted by every group to be another non-example of overlapping polygons (in the vein of the provided non-examples in Figure 1) rather than a $(5,2)$ $(5,2)$ star polygon with more "empty" vertices. When this was the case, technology acted as a *reorganizer* (in the sense of Sherman, 2014) by shifting the focus of participants' thinking. Unfortunately, this reorganization appeared to remove an opportunity for theoremizing and defining.

Looking Forward

Another avenue for future research could be to investigate the didactic potential of the star polygons task itself for student guided reinvention of topics in either abstract algebra or number theory using the instructional design theory of realistic mathematics education (Gravemeijer & Doorman, 1999). Evidence in the star polygon submissions indicates that star polygons may act as an experientially real setting for recreating Euler's totient function, finite cyclic groups, their generators, and isomorphisms between finite cyclic groups of the same order.

Finally, we wonder what other ways this type of scripting task could be leveraged to observe mathematical activity in investigation group work. Future research may not be directly related to star polygons, but instead seek to apply scripting as a means of data collection for other mathematical investigations.

References

Bergman, A. M., Gallagher, K., & Zazkis, R. (2022). Prospective Teachers' Responses to Students' Dialogue on Fractions: Attribute Substitution and Heuristic Approaches. *Research in Mathematics Education*. Advance online publication.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*, 77-101.

Da Ponte, J. P. (2007). Investigations and explorations in the mathematics classroom. *ZDM Mathematics Education*, *39*, 419–430.

Gourdeau, F. (2019). Problem solving as a subject and as a pedagogical approach, and the ongoing dialogue between mathematics and mathematics education. In P. Felmer, P. Liljedahl, & B. Koichu (Eds.), *Problem solving in mathematics instruction and teacher professional development* (pp. 23-42). Springer, Cham.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational studies in mathematics, 39(1),* 111-129.

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics, 93(2)*, 223-243.

Leikin, R. (2014). Challenging mathematics with multiple solution tasks and mathematical Investigations in geometry. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 59–80). Dordrecht, the Netherlands: Springer.

Liljedahl, P. (2007). Persona-based journaling: Striving for authenticity in representing the problem-solving process. *International journal of science and mathematics education*, *5(4)*, 661-680.

Marmur, O., Moutinho, I., & Zazkis, R. (2020). On the density of Q in R: Imaginary dialogues scripted by undergraduate students. *International Journal of Mathematical Education in Science and Technology*, 1-29.

Marmur, O., & Zazkis, R. (2018). Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics, 99(3)*, 261-275.

Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic analysis: Striving to meet the trustworthiness criteria. *International Journal of Qualitative Methods, 16*, 1-13.

Pehkonen, E. (2019). An Alternative method to promote pupils' mathematical understanding via problem solving. *In Problem solving in mathematics instruction and teacher professional development* (pp. 111-122). Springer, Cham.

Perdomo-Díaz, J., Felmer, P., & Rojas, C. (2019). Teachers' Mathematical Tensions Surfacing During the First Session of a Problem-Solving Professional Development Workshop. In P. Felmer, P. Liljedahl, & B. Koichu (Eds.) *Problem Solving in Mathematics Instruction and Teacher Professional Development* (pp. 359-378). Springer, Cham.

Quinnell, L. (2010). Why Are Mathematical Investigations Important? *The Australian Mathematics Teacher, 66(3)*, 35–40.

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical thinking and learning, 7(1)*, 51-73.
Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics, 88(2)*, 259-281.
Rott, B., Specht, B. & Knipping, C. (2021) A descriptive phase model of problem-solving processes. *ZDM Mathematics Education, 53*, 37–752.
Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. *Handbook of research on mathematics teaching and learning*, 495-511.
Yerushalmy, M. (2009). Educational technology and curricular design: Promoting mathematical creativity for all students. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 101–113). Rotterdam: Sense Publishers.
Zazkis, R., & Chernoff, E. J. (2008). What makes a counterexample exemplary?. *Educational Studies in Mathematics, 68(3)*, 195-208.
Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching versus teaching planning. *For the learning of mathematics, 29(1)*, 40-47.
Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). Lesson play in mathematics education: A tool for research and professional development. New York: Springer.
Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education, 16(1)*, 54-70.