Effects of Representations in Complex Analysis on Student Understanding in Real Analysis

Aubrey Kemp
CSU, Bakersfield

Brian Ryals
CSU, Bakersfield

Jonathan Troup
CSU, Bakersfield

Research suggests that providing students opportunities to learn about concepts in a different context can deepen their understanding of these concepts (Kemp & Vidakovic, 2022; Troup et al, 2022; Baker et al. 2000, Dreiling, 2012; Hollebrands et al., 2010). Additionally, recent research has suggested that students reasoning about mappings from $\mathbb{R} \to \mathbb{R}$ rather than graphs in $\mathbb{R}^2$ may open new affordances for students in their reasoning about mappings from $\mathbb{C} \to \mathbb{C}$, as they attempt to generalize from graphs in $\mathbb{R}^2$ rather than mappings from $\mathbb{R} \to \mathbb{R}$ (e.g., Soto & Hancock, 2019; Troup et al., 2017). The proposed intervention investigated in this study represents a potential converse to Troup et al.'s observation in that learning about concepts in Complex Analysis may offer opportunities for student understanding of concepts in Real Analysis to deepen. This study was conceived due to students in one of the researcher's Complex Analysis class expressing that learning about sets and neighborhoods in the Complex plane made aspects of real analysis, such as the $\epsilon - \delta$ definition of a limit, more understandable. The researchers' intent for this pilot study was to determine more precisely the possible ways in which this realization can occur. We hoped to identify, (1) In what ways is student understanding of concepts (such as sets and neighborhoods) in Real Analysis affected by being introduced to these concepts in the Complex space?, and (2) What pedagogical implications emerge from obstacles students may face seeing different representations of these concepts?

In this poster, we outline an ongoing study intended to characterize, through the lens of APOS (Action, Process, Object, and Schema) Theory, ways in which learning concepts in Complex Analysis might render concepts in Real Analysis more approachable for a student. Baseline data was collected via a Real Analysis class assignment in Spring 2022 at a Western Hispanic Serving Institution. We then introduced the students to an 18-minute video mini-lecture covering geometric representations of ϵ and δ neighborhoods in the complex plane and a corresponding definition of limit. We collected data afterward with another class assignment and compared this to our baseline data to determine what, if anything, changed in students' reasoning and responses after watching the video. Our findings from this pilot study revealed some easily correctable limitations of the current methodology, including the format and timing of data collection. Despite these limitations, initial results suggest a common perspective the researchers named *range-axis entanglement* where the student had a strong tendency to believe the inputs and outputs of a function must be graphically represented on an xy-plane. We believe this may be a consequence of an early coordination of Processes within the Limit schema.

We hoped that exposure to these concepts in a different context could offer affordances for students to overcome obstacles and better reason abstractly. In future data collection, we plan to adjust the presentation of the relevant concepts in the context of Complex Analysis as an activity in class instead of a video. We also plan to collect data through interviews to target student understanding in both Complex and Real Analysis. Particularly, that as students become proficient with mappings $\mathbb{C} \to \mathbb{C}$, and in so doing disentangle the range and axis concepts, they might become more proficient with mappings $\mathbb{R} \to \mathbb{R}$ as a resultant side effect. For this poster, we present these preliminary findings, including examples of the range-axis entanglement, from the first stage of our study. We also provide future research plans, including ways we intend to refine the methodology for future stages of our larger project.

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CALIFORNIA STATE UNIVERSITY BAKERSFIELD

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Abstract: Research suggests that providing students opportunities to learn about concepts in a in a different context can deepen their understanding of these concepts. Additionally, recent research has suggested that students reasoning about mappings from $\mathbb{R} \to \mathbb{R}$ rather than graphs in $\mathbb{R}^2$ may open new affordances for students in their reasoning about mappings from $\mathbb{C} \to \mathbb{C}$, as they attempt to generalize from graphs in $\mathbb{R}^2$ rather than mappings from $\mathbb{R} \to \mathbb{R}$. The proposed intervention investigated in this study represents a potential converse to this observation that the lack of opportunities for students to engage with $\mathbb{R} \to \mathbb{R}$ in comparison with $\mathbb{R}^2$ leads to difficulty with mappings $\mathbb{C} \to \mathbb{C}$. In this poster, we outline an ongoing study intended to characterize, through the lens of APOS (Action, Process, Object, and Schema) Theory, ways in which learning concepts in Complex Analysis might render concepts in Real Analysis more approachable for a student. We present preliminary findings from the first stage of our study. We also provide future research plans, including ways we intend to refine the methodology for future stages of our larger project.
Key words: Limits, $\varepsilon - \delta$ proofs, APOS Theory, Visualization, Complex Numbers

Research Questions

1. In what ways is student understanding of concepts (such as sets and neighborhoods) in Real Analysis affected by being introduced to these concepts in the Complex space?
2. What pedagogical implications emerge from obstacles students may face seeing different representations of these concepts?

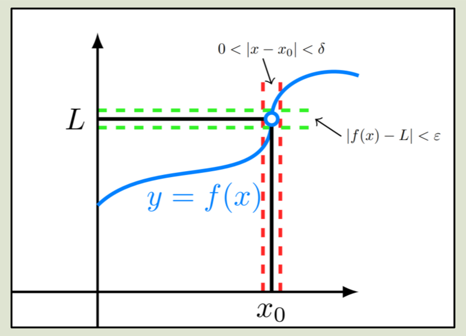
Literature Review

Providing students opportunities to learn about concepts in a different context can deepen their understanding of these concepts (Kemp & Vidakovic, 2022; Troup et al., 2022; Baker et al. 2000, Dreiling, 2012; Hollebrands et al., 2010)

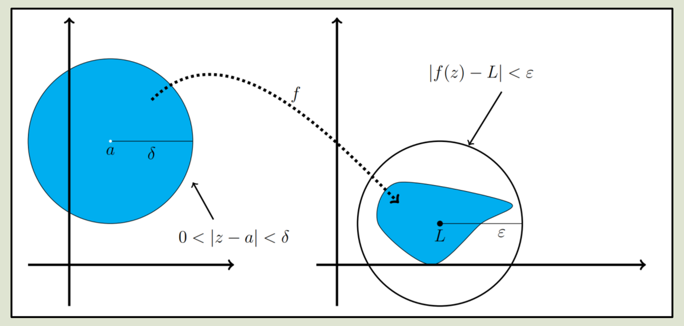
Limits:
- Students rely heavily upon graphs and formulas in understanding the limit concept (Williams, 1991)
- *Obstacle of formalism*: the tendency for students to view the formal representation of a mathematical object as if the representation was itself the object (Sierpinska, 2000)
- In addition to relying on a formal definition of limit, students can benefit from dynamic/geometric ideas and representations of limits (Tall & Vinner, 1981)
- Reasoning about mappings from $\mathbb{R} \to \mathbb{R}$ rather than graphs in $\mathbb{R}^2$ may open new affordances for students in their reasoning about mappings from $\mathbb{C} \to \mathbb{C}$, as they attempt to generalize from graphs in $\mathbb{R}^2$ rather than mappings from $\mathbb{R} \to \mathbb{R}$ (e.g., Soto & Hancock, 2019; Troup et al., 2017)

APOS (Action-Process-Object-Schema) Theory (Dubinksy, 2002; Arnon et al., 2014):
- *Action* – can transform objects with external stimuli or by performing steps
  Ex: For $\lim_{x\to a} f(x) = L$ - may evaluate f at a single point x that is close to or equal to a.
- *Process* – can imagine performing these Actions without necessarily doing so; can *coordinate* Processes with others to form connections between concepts
  Ex: May evaluate the function at more than one point, each point getting nearer to a, interiorize this action, and construct a domain process in which x approaches a. They also may construct a range process in which $f(x)$ approaches L, and coordinate these processes by applying the function f to both processes
- *Object* – can think of a Process as a totality to which Actions or Processes may be applied
  Ex: Considering limits at a point of combinations or compositions of functions

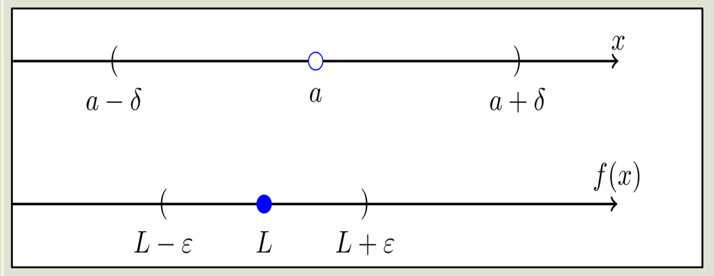
Visualizations



Typical geometric picture in $\mathbb{R}$ (focus on pre-image and image as one graph)



Typical geometric picture in $\mathbb{C}$ (focus on pre-image and image as two graphs)



Desired geometric picture in $\mathbb{R}$ (focus on pre-image and image as two graphs)

Methodology

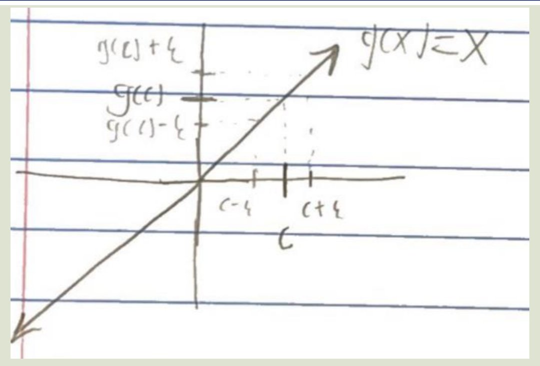
- 13 volunteers from a Real Analysis course at a western university (out of 20 students enrolled)
- Coursework (homework, assessments) collected for analysis
- Participants:
  - Saw traditional classroom algebraic and geometric treatment of one dimensional $\varepsilon - \delta$ limits in $\mathbb{R}$
  - Watched short lecture on the algebraic and geometric treatment of one dimensional $\varepsilon - \delta$ limits in $\mathbb{C}$.
  - Were asked to draw "before" and "after" pictures that correspond to the pre-image and image and to elaborate on their thinking
- Using APOS Theory, analyzed whether there were then any changes in their geometric understanding in $\mathbb{R}$.

Definition Given in Class

4.1.4 Definition Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of** f **at** c if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

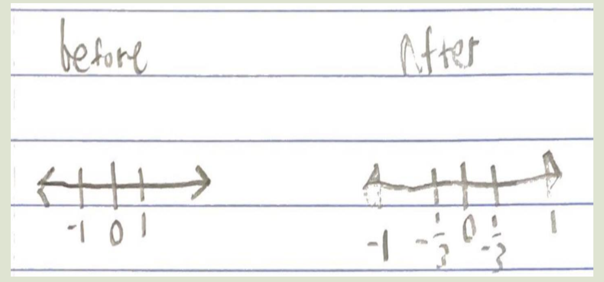
Student 1

Before Exposure to $\mathbb{C}$



"The limit of the polynomial $g(x) = x$ is precisely $g(c)$ for c in $\mathbb{R}$, since if $0 < |x - c| < \varepsilon \to |x - c| < \varepsilon$. In other words, **as we approach the** $\boldsymbol{c}$ **value, we also approach the** $\boldsymbol{g(c)}$ **value.**"

After Exposure to $\mathbb{C}$



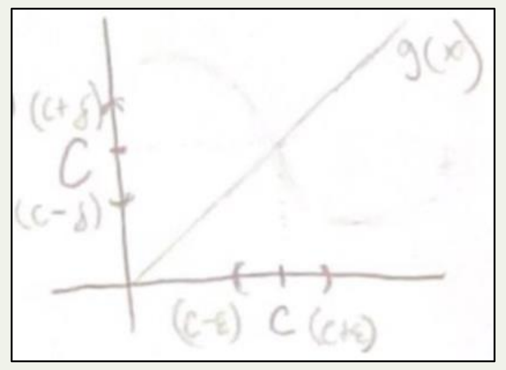
"The function $3x$ **stretches the real number line** by a factor of 3. Since $\dim(\mathbb{R}) = 1$, **our pictures only have one dimension**, whereas $\dim(\mathbb{C}) = 2$ as seen as $\mathbb{C} = \mathbb{R} \times \mathbb{R}$."

Analysis Highlights

- Before: Process conception ("approach")
- After: Process conception (if prompted may show evidence of Object)
- *Limit schema* – Evidence of successful coordination between image and pre-image processes
- No evidence of Range-axis entanglement

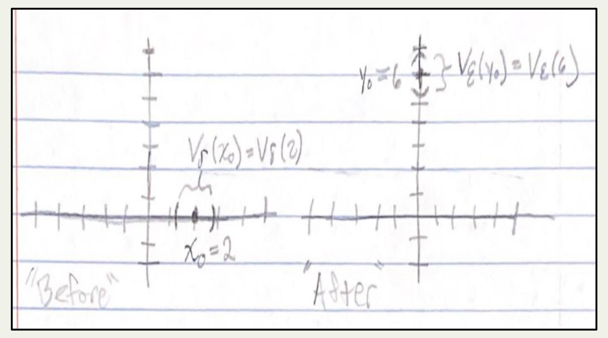
Student 2

Before Exposure to $\mathbb{C}$



"If an **infinite amount of points** is in the ε –neighborhood of c, then there also [is] **an infinite number** of points in the δ –neighborhood of c. So the **number** $\boldsymbol{g(x)}$ **approaches as** $\boldsymbol{x}$ **approaches** $\boldsymbol{c}$ **is also** $\boldsymbol{c}$."

After Exposure to $\mathbb{C}$



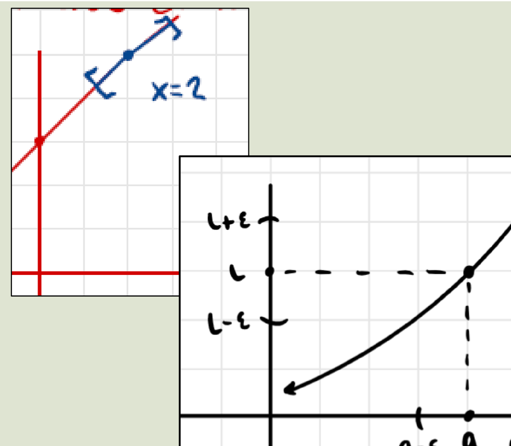
"In this function $\boldsymbol{f(x)}$ **is our output and** $\boldsymbol{x}$ **is our input**. So we can have $f(x)$ be our after image and x be our before image. So if we have an ε-neighborhood around $y_0 = 6$ then we know **there must be a corresponding** δ-neighborhood around $x_0 = 2$."

Analysis Highlights

- Before: Process conception (relating neighborhoods as processes)
- After: Process conception (relating the neighborhoods as objects)
- *Limit schema* – Object conception of **Function**; May need to de-encapsulate **Function** object to strengthen the coordination of **Range** and **Graphing** processes
- Range-Axis Entanglement

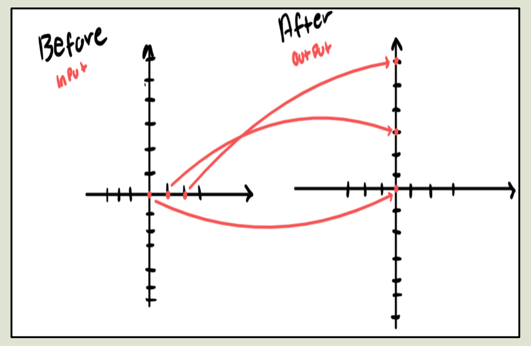
Student 3

Before Exposure to $\mathbb{C}$



"Could be that c is the same distance as epsilon, not sure. Cannot write a full explanation as I don't know where c is.

After Exposure to $\mathbb{C}$



"For the before graphs, I **choose the** $\boldsymbol{x}$ **values** with $x \in \mathbb{Z}$. I stopped at $x = 2$ because $c = 2$. All of these points remain on the x-axis. For the after graphs, **I used the points from the** $\boldsymbol{x}$**-axis and plugged into** $\boldsymbol{y = 3x}$. My $\boldsymbol{y}$ **values only stuck to the** $\boldsymbol{y}$**-axis.**"

Analysis Highlights

- Before: Pre-Action conception (could refine understanding of cluster point)
- After: Action conception (picking specific points to relate)
- *Limit schema* – Action conception of **Function**; May need to work on coordination of **Pre-Image** and **Image** processes to interiorize

"Why Complex?"

- Symbolically, the definitions of a limit in the Real and Complex space are nearly identical, though the visual interpretation is different
- Promotes thinking of neighborhoods and their images as sets
- Detangles the function f from the y-axis
- Visual generalizes to more abstract setting (traditional visual doesn't!)
- Complex still uses same notation as $\mathbb{R}$ (unlike $\mathbb{R}^2$)
- Offer affordances for a better comprehension in $\mathbb{R}$?

Preliminary Results

- Did not clearly affect understanding
- Possibly allowed for coordination reversal for some students
- Illuminated schema development and obstacles within
- *Range-Axis Entanglement*: Strong tendency to believe the inputs and outputs of a function must be graphically represented on an xy-plane (face an obstacle in "separating" pre-image and image)
  - Consequence of early coordination of Processes within the Limit schema?
  - How can we effectively guide students?

Limitations of Present Study

- Format of data collection
  - No interviews to probe for understanding
  - Questions in prompts did not target understanding in both $\mathbb{R}$ and $\mathbb{C}$ as well as they could
  - Video was 18 minutes → In class activity next time
- Timing of data collection
  - Collect after semester or at least not during finals
  - Introduce concepts $\mathbb{C}$ in class before this video

Future Directions

- As students become proficient with mappings $\mathbb{C} \to \mathbb{C}$, and in so doing disentangle the range and axis concepts, do they become more proficient with mappings $\mathbb{R} \to \mathbb{R}$?
- Continuing investigating this idea and *range-axis entanglement*
- Student understanding of absolute value in relation to the limit definition
- How does replacing the notation $|x - y|$ with $d(x, y)$, where d stands for distance, affects student conceptions.

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