Implementing and Scaling-Up of Research-Based Precalculus Curriculum

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Small-scale teaching experiments focused on how students reason about exponential functions provide an incredibly useful foundation when thinking about the design of classroom tasks. However, scaling-up certain critical features of these experiments such as focused and prolonged instruction on a single topic is unrealistic for those teaching at the university level. Drawing on research on covariational reasoning, exponential functions, and design-based research methodology, we developed and implemented two interactive lessons with the goal of enhancing students' understanding of exponentiation and covariational reasoning. In particular, we sought to identify the quantities and relationships that students focused on during classroom tasks and the ways in which the tasks could be refined to support students' understanding of exponential covariation. From this analysis, revisions made to the curricular materials and other implications for research and teaching are discussed.

Keywords: Precalculus, Covariational Reasoning, Exponential Functions

Researchers in undergraduate mathematics have paved the way for the rise in conceptually oriented undergraduate mathematics courses throughout universities in the United States (e.g., Thompson & Carlson, 2017). For many universities, however, implementation of these courses is constrained by several factors such as large class sizes, lack of research-based curriculum materials, and lack of professional development and support for instructors (e.g., Deshler et al., 2015). While many theories exist about instructional means of supporting conceptually oriented undergraduate mathematics learning, there is still a gap in the knowledge of how to scale-up the research findings from these small-scale teaching experiments to instruction and curriculum design in large university courses. This study seeks to fill this gap in the literature by developing and modifying classroom tasks from small-scale teaching experiments to support students' reasoning about exponential covariation in the scaled-up setting of a large undergraduate precalculus course. In particular, we sought to identify the quantities and relationships that students focused on during the classroom tasks and the ways in which the tasks could be refined to support students' understanding of exponential covariation in this scaled-up context.

Literature Review

Quantitative and Covariational Reasoning

Quantitative and covariational reasoning have been shown to be foundational reasoning abilities important for learning about functions and other related mathematical concepts (e.g., Carlson et al., 2002; Oehrtman et al., 2008; Thompson & Carlson, 2017). When students reason quantitatively, they are able to "conceptualize measurable attributes of some object or situation" (Madison et al., 2015, p. 56) and the relationships between these attributes. This reasoning serves as a basis for covariation which entails coordinating how two quantities change in relationship with each other (Thompson & Carlson, 2017). Instruction aimed at the development of quantitative and covariational reasoning has been shown to improve student learning outcomes in undergraduate precalculus and prepare students for success in future mathematics courses such as calculus I (Carlson et al., 2010; Carlson et al., 2015).

Exponential Functions

Historically, instruction on exponential and logarithmic functions has been based on the ideas of exponentiation as repeated multiplication and as logarithms as either repeated division or simply as the inverse to exponentiation (Ellis et al., 2012; 2015; Kuper & Carlson, 2020). Recent studies (e.g., Kuper & Carlson, 2020) have found that students' understanding of exponential and logarithmic functions is greatly improved when they are given opportunities to reason quantitatively and covariationally.

Ellis et al. (2012; 2015) analyzed three 8^{th} grade students' reasoning while working in the context of a magical growing cactus named the Jactus to explore the conceptual shifts that students made to reason productively about exponential growth. These shifts included moving from a repeated multiplicative view of exponentiation to the coordination of constant multiplicative ratios between consecutive output values with constant additive changes in consecutive input values. An example of this is students' coordination of the ratio between the Jactus' heights at different times and the elapsed growth time. In other words, when given the Jactus' height at weeks 1, 2, and 3, students were able to divide the successive heights and coordinate this multiplicative ratio with the elapsed growth time of 1 week to come to the conclusion that the Jactus' height doubles every week. Ellis and collaborators observed students' levels of correspondence and covariational reasoning for repeated consecutive input values greater than a one-unit change preceded their productive reasoning with non-natural exponents. For example, students had to understand how to reason about the ratio between the Jactus' heights at 2 or 3 week intervals (i.e., the Jactus triples every 2 weeks as opposed to every week) before they were able to productively reason about the ratio of heights at 1 day intervals. For example, the students were told the Jactus doubles in height every week and were asked to find its height at 0.25 weeks. Productive covariational reasoning about these non-natural exponents occurred after students were familiar with natural numbers larger than 1 as the exponent.

Methodology

The analysis presented here is part of a broader study aimed at identifying and analyzing the key features of task design and classroom instruction that are beneficial for student learning when attempting to scale-up teaching experiments to whole classroom settings. We begin this work by developing and modifying classroom tasks used in prior teaching experiments (i.e., Ellis et al., 2012; 2015) to support students' reasoning about exponential and logarithmic covariation in the scaled-up setting of a large undergraduate precalculus course. We used design-based research methodology because its purpose is to simultaneously "develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb et al., 2003, p. 10). The strength of this methodology for the purpose of scaling up curriculum and instruction is that we are able to utilize the empirical evidence from studies such as those described in the literature review to analyze and test which features are most crucial for students learning when in the new setting of a large undergraduate precalculus course.

Setting

This study took place at a large research university in the Southeast United States. The participants were 30 undergraduate precalculus students enrolled in a six-week course which included both lecture and lab or recitation sessions. Due to the accelerated nature of a six-week course, students met daily for lectures and twice weekly for lab sessions. During one lab meeting students took a weekly quiz in lieu of unit tests. In the other lab, students worked in small groups of 3–4 on assignments created by the research team which drew on literature such as that described above. The purpose of these labs was to support development of quantitative and covariational reasoning as well as deepen their understanding of key mathematical concepts. For this study, the constraints of this course structure were a crucial part of the design of lab tasks. For example, lab sessions were limited to 60 minutes and the only assistance that students received outside of their small peer group were from their instructor, a graduate teaching assistant, and the first author of this report. Unlike small-scale teaching experiments where one teacher works consistently with 2–3 students, our design needed to account for having two to three instructors working with 30 students during a single lab section.

Data Sources

Students' anonymized written work from the labs were used as the primary source of data analyzed for this study. In total there were six groups of 3–4 students who worked through 6 lab activities. Each lab consisted of a brief pre-lab assignment, a lab activity, and an exit ticket. Additional data that informed our lab revision process and analysis came from field notes taken by the first author during labs and from debrief meetings with the instructor for the course and the research team. Since the focus of this particular study was primarily on students' understanding of exponential functions with an additional emphasis on quantitative and covariational reasoning, we decided to limit our analysis to the lab which introduced these concepts.

The Mathematics. The problem context of the lab was Alice in Wonderland. In the pre-lab activity, students were expected to find the number of sips of potion needed for Alice to fit through the door when given information about her original height and how a potion changes her height as she takes sips. During the lab, students worked through tasks which asked them to calculate the first and second differences in her heights as she sips the potion and to plot the data on a graph. Following this, students had to consider how Alice's height changes when half sips are taken and estimated her height at half-sip values. The second portion of the lab focuses on Alice eating a cake which makes her grow. Students are given a partially-filled table with her height after a number of bites of cake and are asked to fill in the missing height data. Then they are asked to graph their data and to again consider how Alice's height changes when taking half-bites of cake. For this report, we drew on data from two questions (See Table 1) which we felt had the highest potential for supporting students' covariational reasoning.

Table 1. Alice in Wonderland Lab Questions

Problem 2e
Students have the following information: (1) Alice shrinks to ⅔ her current size each time she takes one sip, and one sip is 0.5 oz. (2) Alice was 51 inches tall before drinking any potion, and 34 inches after 1 sip (or ½ oz) of potion. **Students are asked this question:** We know that after her first sip of potion Alice's height decreased by 17 inches. If Alice had instead only taken half a sip of potion (0.25 oz) what is a reasonable expectation for the decrease in her height? Would it have been more than, less than, or equal to 8.5 inches? Explain your reasoning.

Problem 3d

Students have the following information: Alice has eaten 8 bites of cake and we know Alice's height after various bites of cake (e.g., 0 bites, 4.5 inches tall; 2 bites, 8.85 inches; 6 bites, 33.88 inches).
Students have to find the following information: Alice's height after 5 and 8 bites
Students are asked this question: Based on our table of data we know that after two bites of cake, Alice's height increased by 4.32 inches. If Alice had eaten each bite of cake separately would each growth spurt be a 2.16 inch increase in height? Why or why not?

Analysis

Tyburski et al. (2021) argue that covariational reasoning is clearly delineated into levels with associated mental actions involved, but in practice there is little guidance for deciding what moment-to-moment student words or actions are evidence of their level of covariational reasoning. They offer lessons from their work to researchers who want to analyze and make claims about students' covariational reasoning. In the spirit of their work, we began our analysis by drawing on students' written work to identify the quantities that students attended to and evidence of the ways in which they coordinated those quantities. We used this along with the field notes and observations made by the research team to group students' responses into categories we felt were indicative of the types of quantities and covariational reasoning observed.

Findings

The intention of the lab was to support students' reasoning and foundational conceptions about the exponential covariation of quantities. We wanted students to coordinate additive changes in bites/sips with multiplicative changes of Alice's height. Additionally, we wanted students to understand that Alice's height would decrease (or increase) by smaller and smaller amounts (or larger and larger) as she continued to sip potion (or eat cake). In order to get students to think about this, we asked them to consider whether or not Alice's difference in height would be halved when she took a half sip instead of a full sip (i.e., Problem 2e). We anticipated that students would reason about this by noticing that when you multiply by ⅔ (the shrinking rate), that the number shrinks more when you begin with a larger number. In other words, Alice shrinks by larger amounts when she is taller, but her changes in height decreases as she gets shorter. However, we found that the resources embedded in the design of the lab leading up to problem 2e did not provide students with adequate resources to appropriately coordinate Alice's heights and the successive changes in heights with each sip of potion. Indeed, five of the six groups responded incorrected to problem 2e by reasoning in ways such as "In order [for] a full ⅔ decrease, she would have to take a full 0.5 oz sip, but she only took half; therefore the ⅔ decrease would be ½ [as] effective, so she decrease[d] by ⅓, so her height decrease[d] would be 5.66 [inches]," or that "Alice's height [would] decrease by 8.5 inches because 17 divided by 2 equals 8.5." The only group who responded correctly used a previously learned formula (which was not our intention).

In examining student responses to question 3d, however, we found that students had far more ways to reasoning productively about the quantitative relationships involved. Additionally, every group correctly answered that Alice's growth would not be in two equal spurts of 2.16 inches. We grouped the ways of reasoning about this into three (often overlapping) categories as shown in Table 2.

Table 2. Examples of Student Work on Question 3d

Reasoning	Groups	Response
Change is **not** constant	2, 3, 6	"It would not be constant, so she would not grow 2.16 each time, instead the second time would be greater."
Change is multiplicative	2, 5	"No, because we're not adding, we are multiplying."
Change is exponential	1, 4, 6	"No, the rate of change is defined as exponential growth. This means each bite stimulated a greater change in height than the preceding bite."

Discussion

Our findings support those from Ellis et al. (2015) in documenting the conceptual difficulty of reasoning covariationally with exponential functions. We observed a pattern in students' reasoning (and our own when designing and planning the learning trajectory) where one wants to use linear relationships when partitioning both quantities rather than coordinating additive and multiplicative changes. We found that students frequently described changes in height in terms of linear relationships in question 2e (when they had to reason about a partition of a single pair of values). In contrast, students more frequently conceptualized the relationship in question 3d multiplicatively (when they had to reason about successive changes of sips and heights). While it was our intent that both questions supported the coordination of additive changes in one variable with multiplicative changes in the other, students did not reason in the way we intended until they were forced to reason about iterative or successive changes.

In examining other features of the task design, we also noted that the resources available to the students at the time of each problem were starkly different. Prior to question 2e, students were told that Alice's height was 51 inches and that she would shrink to ⅔ of her original height with each sip of potion. Then students were given a table of values filled in with her height at one sip intervals from 0 to 10. In contrast, prior to question 3d students were given Alice's initial height and a partially filled table of values (See Table 1). The students were tasked with figuring out Alice's height given only the information from this table before being asked question 3d. When students had to reason about the quantities and their relationships to fill in the table, it appeared to also support their ability to reason productively in question 3d (where every group answered correctly).

Based on our observations, we refined the content of the lab so that students would have the opportunity to engage in reasoning about the coordination of quantities earlier. We also set up tasks that specifically brought out some of the reasoning from their responses in order to challenge their thinking about constant versus non-constant rates of change. This preliminary report offers one way that researchers might use the findings of small-scale teaching experiments to inform task design in larger classroom settings. Additionally, we can use these findings to further our understanding of when and how conceptual shifts occur for students when learning about exponential functions from the frame of quantitative and covariational reasoning.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352. https://doi.org/10.2307/4149958

Carlson, M., Madison, B., & West, R. D. (2015). A Study of Students' Readiness to Learn Calculus. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 209–233. https://doi.org/10.1007/s40753-015-0013-y

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Students' Reasoning Abilities and Understandings. *Cognition and Instruction*, *28*(2), 113–145. https://doi.org/10.1080/07370001003676587

Carlson, M., O'Bryan, A., & Rocha, A. (n.d.). *Instructional conventions for conceptualizing, graphing, and symbolizing quantitative relationships*.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Deshler, J. M., Hauk, S., & Speer, N. (2015). Professional development in teaching for mathematics graduate students. Notices of the American Mathematical Society, 62(06), 638–643. https://doi.org/10.1090/noti1260

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2012). Quantifying exponential growth: The Case of the Jactus. *WISDOMe Monographs*, *2*, 93–112.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, *39*, 135–155. https://doi.org/10.1016/j.jmathb.2015.06.004

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*, *104*(1), 87–103. https://doi.org/10.1007/s10649-020-09951-6

Kuper, E., & Carlson, M. (2020). Foundational ways of thinking for understanding the idea of logarithm. *The Journal of Mathematical Behavior*, *57*, 100740. https://doi.org/10.1016/j.jmathb.2019.100740

Madison, B. L., Carlson, M., Oehrtman, M., & Tallman, M. (2015). Conceptual precalculus: Strengthening students' quantitative and covariational reasoning. *The Mathematics Teacher*, *109*(1), 54–61. https://doi.org/10.5951/mathteacher.109.1.0054

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42).

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education*. Routledge. https://doi.org/10.4324/9781410602725

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B., & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. *Proceedings of The Twentieth Annual Conference on Research in Undergraduate Mathematics Education*.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Tyburski, B. A., Drimalla, J., Byerley, C., Boyce, S., Grabhorn, J., & Moore, K. C. (2021). From theory to methodology: Guidance for analyzing students' covariational reasoning. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1839–1848).