How Students Reason about Compound Unit Structures: m/s^2, ft-lbs, and (kg·m)/s

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Intensive quantities result from quantitative operations on two or more extensive quantities. As such, their units of measure consist of "compound units." Students regularly encounter symbolically-written compound unit structures that are directly given to them, rather than constructed or developed, such as m/s^2, ft-lbs, or kg·m/s. It is consequently important to understand how students might try to reason about such symbolically-presented compound unit structures, which is the focus of this study. We examined "ways of reasoning" students used to make sense of such units, and describe in this paper five themes that emerged during analysis: (1) decomposing into separate units, (2) treating units as variables, (3) using covariational/ multivariation reasoning, (4) posing a quantification, and (5) bringing in pure math concepts.

Keywords: units, quantities, intensive quantities, compound units, ways of reasoning

Introduction and Brief Literature Review

In recent years, mathematics education has taken seriously the need to attend to students' quantitative reasoning (Johnson, 2016; Smith & Thompson, 2007; Thompson, 2011). While quantities might at first seem to be outside the realm of pure mathematics, such reasoning can be important for mathematical sense-making, deeper understandings, and being able to productively use mathematics outside of the mathematics classroom (Ellis, 2011; Jones, 2015, 2022; Moore, 2014; Moore et al., 2020; Thompson, 1994, 2011). Some quantities, called *extensive* quantities, have been defined as being directly measurable using a simple unit, such as length or volume. Other quantities, called *intensive* quantities, have been defined as not being directly measurable, such as speed or work (Kertil et al., 2019; Schwartz, 1988). In this definition, intensive quantities' units of measure consist of *relationships* between other quantities (Lesh et al., 1988). The way we symbolically write unit structures for intensive quantities, called *compound units*, usually suggests the quantitative operations done to achieve the measure, such as *m/s* suggesting a proportion (and hence division) between distance and time quantities (Thompson, 1994).

Much of the work on intensive quantities has focused on the construction of fairly simple intensive quantities and their units of measure (Bowers & Doerr, 2001; Johnson, 2012; Kertil et al., 2019; Thompson, 1994, 2011), or on how reasoning about rates and ratios is beneficial for mathematical learning (Ärlebäck et al., 2013; Boyce et al., 2020; Byerley, 2019). Yet, students regularly encounter more sophisticated intensive quantities and their symbolic compound unit structures, such as m/s^2, *ft-lbs*, or (*kg·m*)/*s*. Further, because context is often in the service of other mathematical learning goals, these compound units are typically given directly, rather than constructed. To use context and quantities effectively, it becomes critical to understand how students might think about these symbolically-presented compound unit structures. This paper contributes by explicitly examining how students might approach reasoning about these compound units. That is, we did not focus on the process of quantity construction, but rather on how students reason about given compound unit structures. Our research question was: What *ways of reasoning* do students use when presented with symbolic compound unit structures?

Theoretical Perspectives: Quantities, Reasoning, and Quantitative Reasoning

Our examination of students' *ways of reasoning about compound unit structures* is based on our theoretical perspectives on (a) quantities, (b) reasoning, and (c) quantitative reasoning. For quantities, we used Thompson's (1990, 1994, 2011) definition: a quantity is a cognitive entity that consists of an object or system, some attribute of that object or system, and a conceivable way to measure that attribute. The last of these defines the "unit" of measure, and it is common practice, for intensive quantities, to symbolically represent that unit in a way that suggests the relationships and measurement process (e.g., *m/s*). In this way, the symbolic unit structure is not necessarily an intrinsic part of a quantity, but is a widespread way to refer to it. However, more complicated compound units might obscure those relationships, such as using a "square" in m/s^2.

We define a way of reasoning as the "activities that students engage in while solving, explaining, justifying, identifying, and so on" (Hohensee, 2014, p. 136; see also Gravemeijer, 2004; McClain et al., 2000). Thompson (1990, 1994, 2011) defines *quantitative reasoning* as a special type of reasoning, specifically "the analysis of a situation into a quantitative structure" (1990, p. 12). Yet, this definition requires that the person start with some situation and then proceed toward a structure, and perhaps a symbolic representation of that structure (e.g., a formula, a unit, or an equation). We believe that students often might be in positions of needing to reason in the opposite direction, from a more complicated symbolic unit to an idea of what that compound unit means. It is this reverse act of quantitative reasoning we focused on in our study. Thus, we define a *way of reasoning about a compound unit structure* as the mental activities one employs when attempting to make sense of a given symbolic unit expression.

Methods

The data used for this paper comes from a 30-minute task-based interview (Goldin, 1997), in which students were shown symbolic compound unit structures and were asked to describe what they meant. This interview protocol thus mimics how we believe students often encounter unit structures in normal educational settings – without specific direction. Of course, the interview contained the artificial directive to make sense of that unit, meaning students' responses were very likely *in-the-moment* rather than *stable* (Thompson et al., 2014). The tasks were chosen to include compound units for different types of intensive quantities: (a) a rate within a rate, (b) a double-proportion (Thompson & Saldanha, 2003), and (c) a double-proportion within a rate:

- You have probably learned about acceleration, which can have this unit in the metric system [m/s^2 written on paper]. What does this unit mean?
- In science, "work" is defined as a product of force and distance, $W = F \cdot d$. If F is measured in pounds and d is measured in feet, what does this unit for work [*ft-lbs*] mean?
- In metric, "momentum" has this unit [$\frac{kg \cdot m}{s}$ written on paper]. What does that unit mean?

10 students (5 female, 5 male) participated in the interview, recruited from four different first-semester calculus classes at a single large university. The recruitment cite was purposeful (Patton, 2002), so that the students would have had exposure to different types of units in math or non-math classes (confirmed by most students' own statements). Our data analysis focused on identifying students' *ways of reasoning*. Analysis began simply by having each of us independently summarize the students' work on each task. We used these summaries to partition the interviews into bounded "reasoning" episodes, beginning when a student started to reason about the unit in a particular way and ending once that train of thought concluded. The students often had several "reasoning episodes" on a single task. Within each reasoning episode, we qualitatively described the actions the students seemed to be taking, based on what they said,

wrote, and gestured. We then grouped related ways of reasoning into distinct themes (Braun & Clarke, 2006), but also tracked variations within each theme. We were less concerned about normative correctness or robustness, and were mainly focused on identifying themes of *how* students approached reasoning about these symbolic unit structures, when asked to.

Results

We organize our results around the five different *way of reasoning* themes we identified.

Way of Reasoning Theme #1: Decomposing into Separate Units

The first theme involved ways of reasoning based on mental actions that split the compound unit structure into separate units, and imagining quantities associated with each part.

Decomposing m/s². For m/s^2, the students employed this reasoning in several different ways:

> *Student 1*: I think of it more as meters per second [writes *m/s*], speed. And an increase in meters per second, per second. This [points to *m/s*] is a rate, and then this [points to m/s^2] is a rate over time, so a change in rate.

> *Student 6*: Acceleration is a velocity over a given amount of time. And, I'm trying to think if that velocity, which is a distance over time, needs to be the same unit of time. [Later:] My natural thinking would put it [i.e., velocity] in meters over seconds… Then I would say it's [i.e., acceleration] over this amount of seconds as well [i.e., the same time value as velocity].

> *Student 10*: So the difference [i.e., between velocity and acceleration] would be the seconds being squared… Say your velocity is you're traveling 1 meter in 2 seconds [writes "$\frac{1\,m}{2\,seconds}$"]. Then, it [i.e., acceleration] would be 1 meter per every 2 seconds squared or 4 seconds [writes "$\frac{1\,m}{4\,seconds}$"].

Student 1 parsed the unit m/s^2 into a well-defined rate (velocity, *m/s*) and a separate well-defined extensive quantity (time, *s*). While Student 6 parsed it similarly, his issue was wondering if the two *s*'s in *m/s* and *s* represented the same time value. The quantities associated with *m, s,* and *s* seemed to retain extensive features for him (as opposed to combining distance and time into a well-defined rate quantity), such that both *s*'s were seen as the same extensive quantity. Student 10 broke up the unit quite differently, into two extensive quantities associated with *m* and s^2. The s^2 seemed to just signify "time," where its value was simply squared.

Decomposing ft-lbs. While *ft-lbs* can only be decomposed into *ft* and *lbs*, students still had different ways of reasoning about the decomposition, as seen in these two excerpts:

> *Student 2*: Maybe in units of, like, pounds-feet, like, you have this many pounds and you have to bring it 1 foot or 2 feet, or 1-foot or 2-foot or 3-foot increments.

> *Student 9*: It is quite literally the pounds of force times the amount of distance it is. So, if you're pushing on something with 20 pounds of force over 10 feet, that's 200 pound-feet.

The main difference was the relationship imagined between the quantities force and distance. Student 2 fixed one quantity (force) at a certain amount, and thought of it being iterated over the other quantity (distance). By contrast, Student 9 seemed to reason about pounds and feet as, essentially, separate quantities that did not interact other than through a multiplication procedure.

Decomposing $\frac{kg\cdot m}{s}$. There was even greater variety to how students decomposed $(kg\cdot m)/s$:

> *Student 1*: You could figure out, with an object's mass and its speed, you would have its momentum. And that would be, you can compare objects of different sizes and different speeds to each other.

Student 8: So, a kilogram is the weight of something, I guess, right. And meters is how far it can travel, given an amount of time. So, like, if we have something that weighs 100 kilograms, and it goes 10 meters, and let's say 10 seconds, then that would mean momentum is 100.

Student 5: So, weight over time times distance over time [writes "$\frac{kg}{s} \cdot \frac{m}{s}$"].

Student 10: So, this top part [circles "*kg·m*"] is work. I know work equals, because that would be a weight times, or a force times a distance. So, we have work per second.

Student 1 broke the unit into an extensive mass (*kg*) and an intensive velocity (*m/s*). Student 8 broke up the unit into three separate extensive quantities (*kg, m, s*). Student 5 mistakenly broke it into two separate intensive quantities (*kg/s* and *m/s*). Student 10 grouped the unit by its numerator (*kg·m*) and its denominator (*s*), incorrectly associating (as other students did, too) "*kg*" with weight or force. This led him to associate this unit with the rate of change of work in time.

Way of Reasoning Theme #2: Treating Units as Variables

The second theme involved ways of reasoning that essentially interpreted the symbols in the unit structure as though they were algebraic variables instead.

Units as variables, m/s^2. Several students justified for themselves the appearance of *m/s*^{2} using reasoning that cast "*m*" and "*s*" as if they were mathematical variables, such as *x* or *y*.

Student 8: If it's the change in speed over time, then it's change in *m s* over *s* [writes "$\frac{m/s}{s}$"]. So then it's like *s* over 1 [writes "*s*/1" in denominator], and then if we flipped it up like that, then it would be, it'd just be meters divided by *s* times *s*. Or meters times 1 divided by *s* times *s* [writes "$\frac{m \cdot 1}{s \cdot s}$"], so it would be meters divided by seconds squared [adds "= *m/s*^{2}"]. That makes more sense.

Student 8's reasoning reflected algebra work. If we swapped the symbols *m* and *s* for *x* and *y*, the reasoning follows the operation of: $\frac{x/y}{y} = \frac{x/y}{y/1} = \frac{x}{y^2}$. The units' symbols, at this moment, were not being treated as units of measure for quantities, but rather as if they were variables. This result suggests that students may blur the line between symbolic units and quantities' values.

Units as variables, ft-lbs and $\frac{kg \cdot m}{s}$. The students also treated symbolic units as variables to justify to themselves why work had the unit of *ft-lbs* and momentum had the unit (*kg·m*)/*s*.

Student 10: Force is measured in pounds. And distance is measured in feet. Pounds times feet.

Student 3: If you take away kilograms, it's just meters per second [writes "*m/s*"]. And then you're multiplying by kilograms. You multiply by kilograms over 1 [writes "$\frac{m}{s} \cdot \frac{kg}{1}$"], because that's kilograms. And because of math, kilograms times meters over seconds [adds "= $\frac{kg \cdot m}{s}$"].

As before, these symbolic units are described as though being algebraically operated on. Student 10's use of pounds × feet is similar to the operation $x \times y = xy$, and Student 3's reasoning about mass × speed follows the algebraic steps of $(x/y) \cdot z = (x/y) \cdot (z/1) = xz/y$.

Way of Reasoning Theme #3: Using Covariational or Multivariational Reasoning

The third way of reasoning theme appeared to be based on a student having already *decomposed the units* and treated them *as variables*. After doing so, some students employed covariational or multivariational reasoning on the unit structures themselves (see Carlson et al., 2002; Jones, 2022; Thompson & Carlson, 2017). In short, covariational reasoning is the coordination of (usually) two variables changing in relation to each other, and multivariational reasoning is the coordination of more than two variables changing in relation to each other.

Covariational Reasoning, m/s^2. In conjunction with how the students decomposed the unit m/s^2, they used that decomposition to covary the quantities imagined as associated with the parts:

> *Student 7*: You have 1 second, 2 seconds, 3 seconds [draws a line and marks these on it]. And right here [points to interval prior to 1 second], the guy's bookin' it, he's going 10 meters per second. And then right here [points between 1 and 2 seconds] he's bookin', he's going 12 meters per second… Whenever you have meters per second squared, you're changing every time. Like every second, this number [points to 10 *m/s*] is changing.

> *Student 4*: If you were doing s is equal to 2, that's [i.e., s^2] going to be 4, because 2 squared is 4. And meters was 6. And then if we say seconds was 3, so 3 squared is 9. But then it [meters] was 20.

Student 7 had broken the unit up as *m/s* and *s* and described how the velocity value would change for different times. Student 4 had split up the unit as *m* and s^2, and thus covaried between time and distance. As s^2 changed from a value of 4 to 9, the value of *m* changed from 6 to 20.

Covariational/Multivariational Reasoning, ft-lbs. For the *ft-lbs* task, the students struggled to coordinate force (*lbs*), distance (*ft*), and work (*ft-lbs*) all together. Instead, they usually ignored the work value while only coordinating force (*lbs*) and distance (*ft*):

> *Student 1*: I think of it as, if you use a certain amount of force, then it's going to go a certain distance. And if you use less force, it going to go less distance. And if there's a greater force that's necessary to have it moved, then it's going to move less.

However, a couple students did attempt to coordinate all three quantities, as in this excerpt:

> *Student 2*: Because, like your work's going to obviously be less if you have to carry it, like, or move it a shorter distance versus a longer distance. But also it's going to increase with your pounds.

A final important part of the students using covariational reasoning with the *ft-lbs* unit structure is that some of them had a strong tendency to want the unit to reflect a *rate* rather than a double proportion. For example, Student 6 began this task by explaining:

> *Student 6*: I was just thinking that it'd be like, pounds over a given amount of distance [i.e., *lb/ft*]. But that's literally saying the opposite. Because this [$W = F{\cdot}d$] is multiply those two together, while putting it in that fashion of saying pounds over a certain amount of feet is in a sense a fraction.

Covariational/Multivariational Reasoning, $\frac{kg{\cdot}m}{s}$. The momentum context has several potential quantities: mass, distance, time, momentum itself, and possibly velocity or even (nonnormatively) "work", depending on the quantities the students envision. The students similarly used the mental action of fixing certain quantities in order to say how others varied.

> *Student 1*: If we're talking about something that goes, let's say 20 kilogram, going 1 meter per second. That's kind of a heavier object going very slowly. And that's going to have the same momentum as a 1 kilogram object going 20 meters per second.

> *Student 3*: Momentum is completely different for two different masses. Because of the difference in masses. The momentum of an entire planet is way different than the momentum of a marble.

> *Student 2*: So, it's in like units of, to move 1 kilogram moving 1 meter per 1 second. And then that's your momentum. And your increments of momentum you'd be working with… So whatever your number means, that's how many times the momentum, that 1 kilogram over 1 meter over 1 second kind of a thing.

Student 1 had broken the unit into *kg* and *m/s*, and then held momentum constant to see the covariational relationship between mass and speed for a fixed momentum. By contrast, Student 3 held parts of the unit-quantities constant (distance and time) to imagine covariation between mass and momentum. Student 2 fixed each of the individual quantities (mass, distance, time) to "1", and then imagined them being iterated in order to create more units of "momentum."

Way of Reasoning Theme #4: Posing a Measurable Attribute

The fourth general way of reasoning involved imagining some attribute the unit might be associated with measuring. All of the students did so on their own, even though they had not been expressly asked to engage in a quantification process, indicating that imagining a quantity might be a common way for students to try to reason about a symbolic unit structure. Yet, they did not seem to think in terms of *unit → quantity → attribute*, but rather directly from *unit → attribute*, suggesting that students may blur the line between a quantity and its symbolic unit.

Posing an Attribute, m/s^2. The students knew that m/s^2 was a unit for acceleration, but when asked what the unit meant, they associated several "attributes" with slight nuances. These included: (a) speed change, (b) going faster or slower, (c) a distance-time relationship, (d) speed at a specific moment, and (e) speed duration over time. We note that (c), (d), and (e) are normatively incorrect, and (b) is ambiguous, as moving fast does not guarantee acceleration.

Posing an attribute, ft-lbs. When asked what "*ft-lbs*" meant, the students similarly associated several potential "attributes" with the unit. These included: (a) an acting force, (b) a sense of effort, (c) a torque (rotational), and (d) a change in kinetic energy. We note that (a) is normatively incorrect and that (b) is ambiguous, as effort and work are not necessarily equivalent. Also, while Student 9 was correct that *ft-lb* was a unit of measure for a change in kinetic energy, he was incorrect to also equate it to the change in velocity of the object.

Posing an attribute, $\frac{kg\cdot m}{s}$. The students created an even wider variety of attributes that they associated with $kg\cdot m/s$. These included: (a) the strength of a collision, (b) the difficulty in stopping an object, (c) the damage or "hurt" done by an object, (d) the effort to move an object, (e) how far an object wants to move, (f) a mass in motion, and (g) an amount of time to cover a distance. Several of these attributes are well-connected to "momentum," though others are problematic, including an amount of time, a distance, and an effort to move an object.

Way of Reasoning Theme #5: Bringing in Related Mathematical Concepts

The final way of reasoning theme involved the activation and usage of related "pure" math knowledge, by which we mean knowledge pertaining to the abstract world of mathematics.

Using related mathematical knowledge, m/s^2. Several students invoked the derivative concept when reasoning about m/s^2. For example, Student 2 began this task by saying:

> *Student 2*: Acceleration is the derivative of your velocity, which is how fast you're going. How many meters over a second. So, it's how much you're changing your velocity at any given point.

Bringing in the derivative helped Student 2 think about "speed over seconds," which then propelled her toward decomposing m/s^2 into m/s and s (see Theme #1).

Another pure math concept students invoked for m/s^2 was graphs. Student 9 used graphs to provide a reasonable justification for a velocity/time measure leading to the units $(m/s)/s = m/s^2$

> *Student 9*: [Draws a position-time graph:] So, as the slope is the change in position over time, which is exactly what velocity is… By the same vein, if we were to graph a velocity function [draws another graph labelled *v* and *t*]… Finding the slope at any given point of the velocity function gives the change in velocity over time at that moment, which is our acceleration.

Using related mathematical knowledge, ft-lbs and $\frac{kg \cdot m}{s}$. One pure math concept that the students used on occasion in reasoning about the *ft-lbs* and *kg·m/s* units was *factoring*:

> *Student 4*: Let's say there's 6 [*ft-lbs*]. There's some options like 2 and 3, or 6 and 1. So, then, if I travelled 3 feet and for each foot I traveled I applied 2 pounds, then overall the amount of work that was put in, is 6 foot-pounds.

Discussion

This study contributes by examining student reasoning in the "opposite" direction of how it is usually studied in the quantitative reasoning literature. That is, rather than examining the construction of a quantity and its unit (Bowers & Doerr, 2001; Johnson, 2012; Kertil et al., 2019; Thompson, 1994, 2011), we examined students' ways of reasoning when tasked with starting with the symbolically represented unit structure and reasoning from there. We posit that this is a common reasoning activity students need to do if we expect them to couple mathematical understanding with context and quantitative reasoning.

One important implication of our study is to problematize the frequent use of symbolic units for intensive quantities as though they are self-evident. A quantity like acceleration is frequently used in mathematics education to enrich understanding of mathematical concepts (Hitier & González-Martín, 2022; Roschelle et al., 2000; Schwalbach & Dosemagen, 2000; Taşar, 2010), but our study shows we cannot assume students have certain ways of reasoning about the unit structure, m/s^2. Our students exhibited many ways to reason, sometimes leading to normative interpretations and other times not. For example, while some students imagined m/s^2 to suggest a rate within a rate, $(m/s)/s$, others imagined it as suggesting a comparison between distance and time, m and s^2. Further, students struggled to make sense of *ft-lbs*, even though such double proportion units show up in algebra, trigonometry, and calculus (Stewart et al., 2021; Sullivan & Sullivan, 2009; Thompson & Saldanha, 2003), including "*person-hour*" tasks. Some students wanted *ft-lbs* to be a rate instead, *lbs/ft*, while others did not maintain a multivariational image between force, distance, *and* work simultaneously. Some students were also unsure of how to imagine $(kg \cdot m)/s$, either incorrectly associating *kg* with a force quantity, or treating all three as distinct extensive quantities (rather than a product of an extensive quantity and an intensive rate).

Our study sheds other useful light on how students think of symbolic units, as well. Students likely blur the line between a quantity, its value, and the symbolic unit used to represent a unit of its measure. The students regularly treated unit symbols, such as *m*, *s*, or *kg*, as though they were algebraic variables, even performing algebraic operations on them. Students also imagined potential attributes that were directly associated with the unit. Thus, rather than an image of a quantity as consisting of separate-but-related object, attribute, unit of measure, and value, the students seemed to freely blend these all together (and unproblematically, from their standpoint). Understanding these ways that students reason can help instructors be more attentive to what students might think when encountering contextualized quantities and units in learning math.

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