Graphical Resources: Different Types of Knowledge Elements Used in Graphical Reasoning

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In broad terms, much of the research on graphical reasoning can be characterized as focusing on misconceptions, covariational and quantitative reasoning, and graphing as a social practice. In contrast, other research has focused on graphing as a cognitive process, emphasizing the fine-grained knowledge elements related to graphing, with a focus on characterizing ideas students associate with graphical patterns (i.e., graphical forms). This paper moves beyond graphical forms to characterize other categories of fine-grained knowledge – "graphical resources" – that are activated and used in concert when constructing and interpreting graphs. In this study, we identified six categories of graphical resources: graphical forms resources, framing resources, ontological resources, convention resources, quantitative resources, and function resources. We posit that holistically considering different categories of fine-grained graph-related knowledge resources can connect various bodies of research on graphing.

Keywords: graphs and graphing, knowledge resources, student reasoning, graphical forms

Graphical reasoning is a central and critical competency within mathematics and across many disciplinary fields (National Council of Teachers of Mathematics, 2006, 2009; National Research Council, 2012). Much of the education research work to date on graphing has focused on taking one large-scale item at a time and examining it, such as examining misconceptions (Aspinwall et al., 1997; Beichner, 1994; McDermott et al., 1987), covariational and quantitative reasoning (Carlson et al., 2002; Thompson & Carlson, 2017), and viewing graphing as a social practice (Roth & Bowen, 2001, 2003) with community-specific conventions (Moore & Silverman, 2015; Moore et al., 2019). More recently, some researchers have begun to take a finer-grained approach to student graphical reasoning, framing graphical activity as a cognitive process (Elby, 2000; Eshach, 2013; Rodriguez et al., 2019a; Rodriguez et al., 2019b). Fine-grained approaches place emphasis on the productive, intuitive ideas students draw on to guide many aspects of their graphing activity (Rodriguez & Jones, in preparation). In fact, a fine-grained view of knowledge structure may provide a more predictive and explanatory account of students' reasoning.

Rodriguez et al. (2019a) examined one key type of intuitive knowledge, called "graphical forms," which are simple ideas associated with a graphical pattern. As a basic example, a student might associate a graph's steepness with the idea of rate, a graphical form aptly named *steepness as rate*. However, this work has focused only on a sub-category of graphical knowledge elements, namely intuitive associations with the graph (curve) itself. In our work, we began to realize there were other types of such knowledge, beyond only graphical forms, making up a significant portion of students' graphing activity. These resources also seemed related to other phenomena identified in the literature, such as misconceptions, covariation, and conventions, suggesting that such fine-grained knowledge plays a significant role across all of students' graphical reasoning. We consequently directed our attention to the research question: What are various types of knowledge resources students use when constructing or interpreting graphs?

Theoretical Framing

We align with the *knowledge in pieces* (KiP) paradigm (diSessa, 1988; diSessa & Sherin, 1998), which moves away from viewing knowledge as consisting of large unitary objects toward viewing knowledge as a complex system of individual elements (diSessa et al., 2016, p. 30). We call such individual knowledge elements “resources” (Hammer, 2000). Resources can be very simple and intuitive, such as diSessa’s examples from physics of *force as mover* or *more effort begets more result* (1988, p. 53). These resources can work together to direct thinking, and the context directly influences which resources are employed (diSessa, 2004). That is, not all resources associated with a given idea are activated in all situations (diSessa & Wagner, 2005). In fact, KiP would view an important part of gaining expertise as reorganizing resources and learning to appropriately apply them in the right contexts (diSessa, 2002).

Sherin (2001) applied the KiP paradigm to students’ understanding of equations and mathematical expressions. He called resources that pair a conceptual idea with a specific symbol structure a “symbolic form.” Some examples are pairing the structure [] + [] with the idea of *parts of a whole*, or the structure [] – [] with the idea of *competition*. Rodriguez et al. (2019a) then took the idea of symbolic forms and extended it to graphs, called “graphical forms”, with a similar idea of pairing a conceptual idea with a specific aspect of a graph. Some graphical forms identified in the literature are *steepness as rate*, *straight means constant rate*, *curve means changing rate*, and *intersection means same* (Elby, 2000; Rodriguez et al., 2019a; Rodriguez et al., 2019b; Rodriguez et al., 2020). A longer library of 26 graphical forms can be found in (Rodriguez & Jones, 2021), and include forms related to points (e.g., *point as instance*), cardinal directions (e.g., *horizontal as constant value*), trends (e.g., *plateau as leveling off*), smoothness (e.g., *curved means realistic*), local features (e.g., *jump discontinuity means sudden*), position (e.g., *displacement as difference*), and pairs of graphs (e.g., *transformation as same*).

Yet, graphical forms may not be the only type of fine-grained knowledge resource students use. In fact, we believe various types of small-scale resources may relate to many of the larger-scale phenomena seen in prior research. In our work documenting graphical forms, we began to identify other types of resources that seemed important for how students constructed or interpreted graphs. This paper contributes by presenting these other types of resources (which we generally call “graphical resources”) and attending to how our graphical resources categories relate to other research on graphing, including misconceptions, covariation, and conventions.

Methods

To examine students as they constructed and interpreted graphs, we created two separate task-based interviews (Goldin, 1997). In the first interview, students were given four different contexts involving real-world quantities and were asked to construct a graph of one quantity as a function of another. In the second interview, students were shown various graphs representing mostly real-world contexts and were asked to discuss what the graphs indicated about the contexts. Figure 1 (top of the next page) contains very brief summaries of the tasks used in the two interviews. Some graphs were taken from or based on a common calculus textbook (Stewart et al., 2021), and others were taken from or based on graphs used in past research (Carlson et al., 2002). Two of the “construction” tasks used time-based graphs, while two used non-time-based graphs because of the research showing students’ potential overreliance on time (Bowen et al., 1999; Jones, 2017; Rodriguez et al., 2020).

We interviewed students in pairs to permit discussion between them and to help make their thinking visible. We recruited 12 students from two different universities, resulting in six pairs. All pairs participated in both interviews, with the two interviews separated by a few days. We recruited students from first-semester calculus courses, to ensure they had familiarity with graphs and graphing (Patton, 2002). We gave the students pseudonyms that suggest their pairings: Anna/Aria, Berto/Blaine, Cindy/Caleb, Donato/Demyan, Ellie/Eric, and Fiona/Felicity.

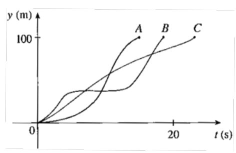
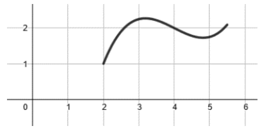
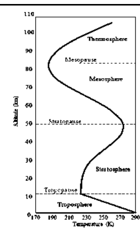
Brief Summaries of Interview 1 Tasks (Construction)			
1. Graph temperature of a frozen pizza that is thawed and baked as a function of time.	2. Graph the grass height of a lawn as a function of time over the summer.	3. Graph the volume of soda in a bottle as a function of the soda height.	4. Graph the distance a puck would travel as a function of the surface's friction.
Brief Summaries of Interview 2 Tasks (Interpretation)			
1. Graph of runners in a race over time. 	2. Graph of position of a remote controlled car over time. 	3. Graph representing temperature at different elevations. 	4. [Various mathematical graphs from a textbook were given. Students were asked to compare and contrast them.]

Figure 1. Brief summaries of the tasks given to students in the two interviews

We analyzed the data as follows. We first divided the interviews into "units of meaning" (Campbell et al., 2013) based on the focus of the students' statements, writing, gestures. This is distinct from dividing interviews based on speaking turns or pauses because it retains the focus on keeping a student's explanation intact. These units were bounded based on discussion turning to a different idea or moving in a different direction. Within each bounded episode, we identified candidate knowledge resources. We inferred these resources by what the student said, gestured, and wrote, while keeping the context of that bounded episode in mind. Our focus was on inductive thematic saturation (Saunders et al., 2018), which emphasizes the generation of new codes. Thus, we were initially very inclusive regarding the list of resources we created. Once we had generated this large list of potential resources, our next analytic action was to refine this list by identifying resources that were largely the same (and combining them) or occasionally recognizing that a candidate resource actually contained two separate ideas (and separating them). We then took our refined list and created larger-scale themes (Braun & Clarke, 2006) that categorized the final list of resources into different graphical resource types. Because the category *graphical forms* already existed, we used that category as an a priori category, but five other categories emerged inductively through our analysis, for a total of six categories.

Results

The six categories of graphical resources we identified were: (1) *graphical forms resources*, (2) *framing resources*, (3) *convention resources*, (4) *ontological resources*, (5) *quantitative resources*, and (6) *function resources*. Because *graphical forms* have been discussed in detail in the literature (see Theoretical Framing section), we do not focus on them in this paper.

Framing Resources

The first category, called *framing resources*, involved resources that paired a simple idea with any aspect of a visual graphical image that was not the graph curve itself, such as the axes, the origin, or blank sections. *Framing resources* are important because they reflect ideas related to aspects of the representation that cue the reader into the context of interest, while also physically encasing (or literally "framing") the graphical curve. To illustrate a few resources in this category, consider Cindy and Caleb, who had started the pizza temperature context by drawing axes. Caleb at first labelled the major waypoints of being thawed, baked, cooled off, and placed back in the fridge (Figure 2). However, Cindy countered with the following:

Cindy: I don't know if this is the best way to represent it, but I feel like you should have some way to represent that the intervals of time aren't the same in between thawing [Th], baking [B], and other stuff like that [C and F].

Figure 2. Caleb's uniform spacing that Cindy felt indicated equal times

Here, Cindy expressed that the equal spacings on an axis indicated equal amounts of time. We call this resource *spacing means duration*. To illustrate another framing resources that, by contrast, dealt with the origin, consider Anna and Aria, working with the same pizza context, who were discussing where to start their graph.

Aria: I just feel, I like starting at the corner [i.e., the origin] … When I start at zero [i.e., the origin], I know that I can like, just go from there. I see it as a starting point.

Aria appeared to draw on a resource we call *origin as starting point*. Interestingly, she and Anna later decided they did not know enough about the Celsius scale (which has freezing point at 0ºC), and chose to use a Fahrenheit scale instead. In this case, rather than starting at the origin, they started at a point on the *y*-axis, in a similar resource we call *y-axis as starting point*.

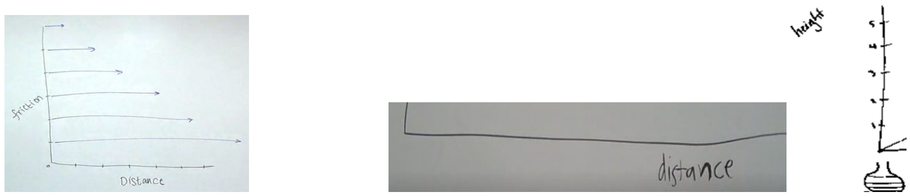
Overall, we found many of these *framing* graphical resources, including some we called *axes need labels, axes need scaling, squiggle means skip*, *y-axis maps to key moment, negative means backward, "+/–" as opposites,* and *axis arrows as continuation*.

Two Types of "Belief" Resources

In our analysis looking for fine-grained resources students used, we identified two separate categories of intuitive "belief-based" resources, regarding implicit assumptions about graphs.

Ontological Resources. The first category of belief-based graphical resources dealt with intuitive beliefs about the very nature of graphs themselves. Thus, we chose to call these *ontological resources*. In fact, one major item in the graphing literature, *graphs as pictures* (Beichner, 1994; Elby, 2000; McDermott et al., 1987), could be reinterpreted as a particularly common and oft-used ontological resource. Here, students believe a graphical image should reflect a picture of the situation. Relevant to our work, we assert that characterizing this as a stable misconception does not adequately capture nuances in students' reasoning and the ways in which this could reflect productive decisions in certain contexts for constructing graphs. Figure 6a below, which is the graph Anna and Aria initially produced for the friction-distance task, shows an example of this ontological resource in action. They claimed that each arrow was one sliding puck at a given friction level. For these students, this reflected an accessible graphing approach that visually resembles the physical scenario while illustrating the expected observation that higher friction (vertical axes) is associated with a shorter distance (horizontal axes).

However, our analysis also revealed a softer version of this ontological resource, in which a graph preserves only a specific visual stimulus. For example, Figure 6b shows the placement of "distance" in the friction task on the horizontal axis and Figure 6c shows the placement of "height" in the bottle task on the vertical axis. While these are certainly fine, mathematically, it did conflict with other student beliefs (see next section) that the independent variable needs to be on the horizontal axis. This result suggests that maintaining visual similarity between the graph and the physical scenario may have a particularly high cueing priority (see diSessa, 1993).



(a) (b) (c)

Figure 6. (a) graph as picture; (b) and (c) preservation of key visual stimulus

While *graphs as pictures* and the softer *graphs preserve visual features* were common, they were not the only ontological resources. For example, when Cindy was interpreting the graph in the atmosphere task, she explained,

> *Cindy*: You can kind of narrate the graph. So, you could be like, as you go up in the troposphere then the temperature… steadily decreases … And then once you get to the stratosphere, then it stays relatively the same for about 10 kilometers…

Cindy drew on a ontological resource we call *graph as a story*. Students held other ontological resources, too, such as *graphs as collections of points*, *graphs as connect-the-dots*, or *graph continuation* (which dealt with the assumption that graphs persist outside the viewing window).

Convention resources. The second type of "beliefs-based" resource were more about fine-grained beliefs about what students considered appropriate graphical practice. That is, rather than being about the *nature* of a graph, these dealt with common social practices (i.e., conventions) (see Moore et al., 2019). Thus, we call these *convention resources*. In fact, based on our work, we believe that much of the convention literature could be reframed to fit within a KiP paradigm. In the following excerpt, Ellie and Eric were working on the bottle task and explained their choice to put volume on the vertical axis and height on the horizontal axis.

> *Eric*: From like the reading of it, 'cause it says as a function of the height. That makes me assume that the height is your independent variable, which is on your x-axis [points to horizontal axis] and your volume is on your y-axis, because it's the dependent variable.

This excerpt contains evidence of three different convention resources. The first is a very common one already described in the literature, where it is believed the independent variable must be on the horizontal axis (Moore et al., 2014). A second related resource is that the *other* variable (ostensibly, the "dependent" variable) must be on the vertical axis. A third, and new, convention resource seen in this excerpt is that the horizontal axis *is* the "x-axis" and the vertical axis *is* the "y-axis." That is, "x" and "y" are essentially the names of the two axes.

Other convention resources dealt with axis direction and scaling. In the pizza task, Anna said,

> *Anna*: Time's going to increase this way [slides hand rightward along horizontal axis]. Temperature, I'm assuming we're going to say it increases [slides hand upward along vertical axis].

Here we can see the simple intuitive idea that *right means increase* along the horizontal axis and *up means increase* along the vertical axes. Note that this is not a graphical form, because the intuitive idea is not coupled with the graph itself, but is rather about the axes. Also, we call these convention resources, because it is not necessary for those directions to mean "increase." In fact, during the friction task, Caleb decided to have the horizontal axis represent *decreasing* friction:

> *Caleb*: 'All' friction to me is zero (i.e., at the origin) … It's weird, I know. But regardless … 'all' friction to me is zero. To me, the *x*-axis is the amount of friction that is *not* present.

The convention resource Caleb appeared to draw on was *direction is relative*, allowing him to decide that he could control which direction meant "increasing." Thus, convention resources can at times involve a recognition of the malleability of conventions (Thompson, 1992).

Overall, we identified many convention resources, including *π implies trig*, *origin is relative,* and *graphs read left-to-right*. Again, we propose that the convention literature might be considered a subset of graphical resources, in that these conventions are often simple, intuitive, and even implicit, ideas students associate with a graphical system.

Quantitative Resources

Another category of graphical resources related to quantities' relationships with contextualized graphs, which we call *quantitative resources*. These resources may be specific to graphs that represent real-world situations. While there were fewer resources in this category, they seemed quite impactful to students' thinking. To illustrate, students had particular ideas associated with what counted as "independent," influencing how they set up their graphical axes. When explaining why time was the independent variable in the pizza context, Anna explained,

> *Anna*: Like, time is going to march on no matter what happens.

We call this *independent as marching on*. The students' work often suggested this resource's use. For example, the bottle task stated height as the independent variable, leading both Eric and Caleb to inaccurately claim that height increase was consistent (or "constant" in their words).

> *Eric*: The height is the constant increase.

> *Caleb*: The height will always increase at the same pace.

In this context, volume might actually be considered the more steadily increasing quantity. This resource actually led several students to want to put the volume on the horizontal axis, even though it conflicted with their resource of "function output" being on the vertical axis.

This resource was strong enough that some students wanted certain graphs to be with respect to time, even when they were not. We call this *time as natural independent*. For example, In the bottle context, Anna and Aria were trying to make sense of height as an independent variable.

> *Anna*: It's weird. 'Cause it's not over a function of time like we've had in the last two graphs … Time feels way easier to me, but that's just me.
> *Aria*: …Time's pretty good and constant, in a sense … time goes constant.

Even when students did use non-time independent variables, they often spoke of it as "marching on" in real time, such as Blaine saying in the bottle context, "We're pouring in a constant volume at each interval of time."

Function Resources

Due to space constraints, we end our results by simply stating there was an additional category of graphical resources, *function resources*, though these figured into the students' reasoning less significantly than the others. Function resources were evidenced when students used explicitly function-based ideas to shape their graphical thinking. Some resources here were *input-output pair*, or *recall of graph shape*.

Discussion

Our contribution has been to expand the work on the fine-grained knowledge resources that students use when constructing or interpreting graphs. We identified six categories of resources students employed in their graphical thinking. This work is important, because these small-scale ideas tended to heavily influence how the students operated graphically, including where to start, how to put pieces of the graph together, how to put surrounding features together (e.g., axes), and what a graph is meant to convey. Collectively, an emergent combination of these graphical resources accounts for a lot of observed graphical activity. In fact, we claim these fine-grained graphical resources may underlie a much what has been discussed in the graphing literature.

For example, consider the work on misconceptions (e.g., Beichner, 1994; McDermott et al., 1987), in which the inferences students draw from graphs are often seen as larger, unitary mental objects. This perspective provides little explanatory and predictive power related to students' reasoning and how instruction can support students (Elby, 2000). Suppose a student sees a straight-line graph with axes that use a logarithmic scale. If a student thinks that the quantities vary linearly, a misconception perspective might posit the student does not understand logarithmic scales. However, it is possible this student is simply invoking the intuitive notion of *spacing means duration*, an assumption that has "worked fine" in the students' past experience (diSessa, 1988). There may be no stable misconception to replace; rather, the student may simply need to be redirected to the fact that *spacing* means something different in this context.

Next, consider the work on covariation (e.g., Carlson et al., 2002; Johnson, 2015; Moore et al., 2013; Thompson & Carlson, 2017). We believe our fine-grained approach can add clarity on how students go from covariational reasoning to a produced graph. That is, much of what is described in the covariational literature is fundamentally *quantitative*. The ideas of one variable increasing as another variable increases, or of tracking the changes in their values, are not inherently graphical. We posit that it takes these fine-grained graphical resources to translate the *quantitative* nature of covariational reasoning into a graphical inscription.

Finally, some important work has been done framing graphing as a situated social practice (Roth & Bowen, 2001, 2003) with community conventions (Moore & Silverman, 2015; Moore et al., 2019). We believe social practices are comprised of many small-scale beliefs that can be flexibly invoked, or not, given a certain context. For example, a student can hold ontological assumptions such as *graph preserves visual stimulus* and *graph as a story* simultaneously. Rather than one of these defining how a student operates for all graphs, they are both fine-grained beliefs that might be activated at different times, depending on the context. Furthermore, we note that some of the graphical resources related to features like the axes (e.g., *axes need labels*) reflect assumptions or beliefs about the nature of graphs or conventions related to graphical activity. This is important because of the way in which these simple ideas influence students' choices about how to proceed with a graphing task (e.g., convention resources such as *independent on horizontal* or *up means increase*).

Together with the previously described graphical forms, the various categories of graphical resources we have described in this paper can greatly enhance our ability to identify the knowledge and ideas students are using when creating or interpreting graphs. We assert this has the potential to serve as a foundation and a unifying perspective that incorporate the many different perspectives used in the graphing literature.

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