Foregrounding Inversely Proportional Relationships in Integration by Substitution

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Background

Foci of Past Calculus Research[1]

- Co-variation
- Functions
- Limits
- Derivatives
- Integrals

Significant Omission

- Students' understandings of multiplication and division *with quantities* in the context of calculus topics

Why it Matters

- College students often have limited understandings of multiplication and division *with quantities*
- Multiplication and division *with quantities* foundational for derivatives, integrals, families of functions central to calculus
- Studying formulae alone unlikely to help students understand quantitative foundations for calculus

The Study

Developed Novel, Semester-Long Course

- Introduced an explicit meaning for multiplication based in measurement

N	•	M	=	P
# of units in 1 group		*# of groups in 1 product amount*		*# of units in 1 product amount*

- Emphasized drawings of lengths and areas
- Worked only with piecewise linear functions and step functions. (No discussion of limits)
- **Strand 1**:Division, prop. relationships, slopes of lines, function composition, chain rule
- **Strand 2**: Multiplication, areas, inversely prop. relationships, integration by substitution

Data Collection

- 18 undergrad. students, Fall 2021
  - 15 completed at least 1 calculus course
  - No STEM majors
- In-class group work on sets of word problems
- Whole-class discussion focused on how the $N \bullet M = P$ structure fit problem situations
- Collected written homework and exams

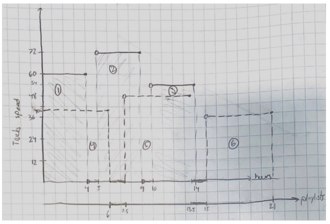
1. Frank, K., & Thompson, P. (2021). School students' preparation for calculus in the United States. *ZDM-Mathematics Education*, *53*, 549–562.
Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium of research in mathematics education* (pp. 526–550). National Council of Teachers of Mathematics.

Results

Strand 2

- Students debated whether rectangular areas could represent products other than areas. For instance, can areas represent distances (e.g., rate • time = distance) or volumes (e.g., area • height = volume)? This is NOT obvious to all students.
- Students were able to reason appropriately about situations in which P was fixed.
- One final exam 13 students gave appropriate solutions to the following problem:
  - Three friends on a road trip. **Steve** drives 60 mph for 4 hrs. They rest 1 hr. **Chara** drives 72 mph for 4 hrs. They rest 1 hr. **Zawadi** drives 54 mph for 4 hrs.
  - Graph speed vs. time. Explain how the graph shows distance each friend drives.
  - Consider same situation but time is measured in playlists; 1 playlist is 40-min.

One Student's Solution



"Area 1 correlates to the distance driven by Steve…. N • M = P ....

60 miles in a column • 4 column = 240 miles

….The new function is drawn with a dotted line….the distance traveled is still represented by the area underneath the function….Area 1 = Area 4….

40 miles in a column • 6 column = 240 miles"

Comments

- A special case of integration by substitution, stated as

$$\int_{g(a)}^{g(b)} f'(u)du = \int_{a}^{b} f'(g(x)) \bullet g'(x)dx$$

- This formulation treats integrand as rate-of-change, consistent with the FTC.
- Justifies equation by explaining that each rectangle in a Riemann sum for the left-hand side has the same area as the corresponding rectangle in a Riemann sum for the right-hand side