Modeling Mathematicians' Playful Engagement in Task-Based Clinical Interviews

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Theoretical accounts of mathematicians' disciplinary practice draw similarities to mathematical play, in that it can entail agency and autonomy, open exploration, creativity, imagination, and enjoyment. We analyzed task-based clinical interviews of 13 mathematicians to determine whether their problem-solving activity was playful and found evidence of three aspects of playful math: agency in exploration or goal accomplishment, self-selection of mathematical goals, and a state of immersion, investment, or enjoyment. We present these findings and introduce a form of problem-solving activity, the Explore-Focus Cycle, as one characteristic of playful math.

Keywords: mathematical play, clinical interviews, mathematicians

One goal of undergraduate mathematics education is to support students' development of productive dispositions towards mathematics (Code et al., 2016; Watson, 2008) helping students engage with mathematics in disciplinarily authentic ways (Lim & Selden, 2009). Work examining mathematicians' practices has highlighted activities such as detecting and formalizing patterns, developing and refining definitions, thinking structurally and generally, selecting and applying proof techniques, and refining earlier reasoning (Burton, 2001; Fernández-León et al., 2021; Melhuish et al., 2021; Martín-Molina et al., 2018). Although it is important to not take an overly narrow view of what it means to meaningfully engage in mathematics as an expert (Stillman et al., 2020), researchers nevertheless argue for instruction that supports the habits of mind believed to be similar to mathematicians' practices, such as generating interesting questions, experimenting, conjecturing, and generalizing (e.g., Bass, 2008; Cuoco et al., 1996).

Gresalfi and colleagues (2018) note that the above forms of disciplinary engagement involve many of the same features as play including "exercising personal agency to set and reach goals, exploration, imagination, and joy" (p. 1335). Others agree: For instance, Mason (2019) remarked that "playfulness provides an accurate and apt description of what mathematicians do," and Su, in his MAA Presidential Address at the Joint Mathematics Meetings (2017), stated, "We play with patterns, and within the structure of certain axioms, we exercise freedom in exploring their consequences, joyful at any truths we find." He went on to identify play as a key component of mathematics for human flourishing (Su, 2020). Indeed, offering opportunities to develop a mathematically playful disposition can center student voices, invite agency, and increase equitable access to mathematics (Gresalfi et al., 2018; Widman et al., 2019).

These remarks reflect a robust belief that mathematicians are playful in their engagement, and yet, we have found little empirical evidence demonstrating these claims. Consequently, the motivation for this study is to investigate whether and how mathematicians' disciplinary engagement is playful. In particular, we consider the following research questions: Is mathematicians' problem-solving activity in task-based clinical interviews playful? If so, what are the characteristics of mathematicians' playful engagement?

Theoretical Framework and Background Literature: Defining and Characterizing Mathematical Play and Playful Math

Definitions of play vary, but all entail a sense of agency in (a) choosing to engage in an activity, (b) exploring, (c) selecting goals, and (d) accomplishing goals (Jasien & Horn, 2018). Huizinga (1955), one of the early scholars to define play, remarked that play is voluntary, steps outside the ordinary, is bounded in time and space, is orderly, is governed by rules, and involves tension; the player wants to achieve something but wants to succeed by their own expertise. Dewey (1916/1966) similarly noted that people who play "are trying to do or effect something" (p. 203). Thus, play is goal-driven, but as Featherstone (2000) noted, the goals of play, unlike the goals of work, "are always in some sense 'inside' the play, they are real and shape the activity of the players" (p. 15).

Mathematical play is a particular type of play. Holton et al. (2001) described mathematical play as the playful exploration that emerges when learners find themselves in mathematical contexts with open goals, and Williams-Pierce (2019) defined mathematical play as voluntary engagement in cycles of mathematical hypotheses with occurrences of failure. Much of the work on mathematical play investigates the mathematics that can arise from playful settings such as preschool children's block play (e.g., Brown, 2009), or players' interactions with videogames (Williams-Pierce, 2019). In contrast, our interest is in how to incorporate play into the school mathematics that occurs in classrooms. We use the term "playful math" rather than "mathematical play" to highlight the potential of "playifying" classroom mathematical activity. Following the traditions of researchers who emphasize agency, we define playful math as a form of mathematical engagement that entails (a) agency in exploration or goal accomplishment, (b) self-selection of mathematical goals, and (c) a state of immersion, investment, and/or enjoyment.

Characteristics of Mathematical Play

Characterizations of mathematical play are typically either situated in studies of young children's activity or in studies of individuals in informal environments. When they concern the mathematical play of undergraduates or experts, characterizations are often, but not always, theoretical rather than empirical. This is likely due to the fact that when students enter high school and college, discussions of learning seldom reference play (Gresalfi et al., 2018). Indeed, play has increasingly been pushed out of school settings (Jerret, 2015), and undergraduate education in particular has seen a separation of play and learning (James & Nerantzi, 2019). Consequently, little is known about the nature of learners' playful engagement in advanced mathematics tasks (Gresalfi et al., 2018).

Many scholars have called for supporting and legitimizing play in adult lives (e.g., James & Nerantzi, 2019; Sicart, 2014, Swank, 2012). Concerning playful mathematics in particular, researchers argue for the importance of understanding and fostering students' playful dispositions as a way to encourage engagement that mirrors the activity of mathematicians. For instance, Featherstone (2000) noted a number of similarities between play and the discipline of mathematics itself in that both focus on theoretical entities, are set apart from daily life, create order, and are rule bound. Mason (2019) explained that "mathematicians play with what is given and with what is sought – extending, varying, simplifying, and complexifying" (p. 95), and other mathematicians have described their own practice as playful (e.g., Lockhart, 2009; Su, 2017).

Descriptions of mathematicians' playful engagement highlight the presence of two phases of activity. Su (2020), for instance, distinguished between the *inquiry phase* and the *deductive reasoning phase*. In the inquiry phase, one engages in pattern exploration and uses inductive reasoning to build conjectures from specific examples. One shifts to the deductive reasoning phase when a conjecture has been selected and must now be proved or disproved. Holton et al.'s (2001) theoretical analysis of mathematicians' play makes a similar distinction between two phases. In the first, one produces tentative examples without a clear direction. In the second, one becomes more systematic and intentional in generating particular types of cases. These phases are reminiscent of Cai and Cifarelli's (2005) examination of undergraduates' mathematical exploration with open-ended problems. Although not specifically referencing play, they distinguished two types of goal episodes, *primitive* and *refined*. Primitive goal episodes entail trying out preliminary ideas without a specific goal other than to generate empirical results to support further exploration. Refined goal episodes are ones in which the problem solver's activity occurs with a well-defined goal in mind.

Although each of these researchers use different terms, all emphasize two characteristics of the first phase: It is empirical, gathering data to inform the development of a pattern, hypothesis, or direction, and it is not goal oriented, in that one is exploring without pursuing a particular, pre-determined direction. The second phase entails goal-directed activity towards a particular outcome in which one's actions are guided by anticipation. There is some disagreement about whether mathematical play is still happening when one shifts into the goal-directed phase (Holton et al., 2001), but there is broad agreement that mathematicians' playful engagement includes open exploration. We therefore attended to these two phases of activity and sought to determine whether mathematicians demonstrated instances of them in their problem-solving activity as a potential characteristic of playful engagement.

Methods

Our data set consisted of task-based clinical interviews from 13 mathematicians (see Lockwood et al., 2016 for additional details). Twelve mathematicians held a PhD in mathematics, and one held a PhD in computer science. During the interviews, the mathematicians worked on the problems shown in Figure 1. Task 1 was taken from Andreescu et al. (2007), and Task 2 was taken from a Putnam exam. All thirteen mathematicians worked on Task 1, and seven also worked on Task 2. We gave the mathematicians time to work on the tasks and asked them to think out loud as they worked. The interviewer generally only interrupted to ask clarifying questions and to answer the mathematician's questions. Each interview lasted approximately 60 minutes and was audio recorded using Livescribe pens and transcribed.

Interesting Numbers Task
Most positive integers can be expressed as a sum of two or more consecutive integers. For instance, 24 = 7+8+9 and 51 = 25+26. A positive integer that cannot be expressed as the sum of two or more consecutive positive integers is therefore *interesting*. What are all of the interesting numbers?

Selfish Sets Task
Define a selfish set to be a set which has its own cardinality as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Figure 1. The Interesting Numbers Task and the Selfish Sets Task.

In analyzing the data, we began with a priori codes based on our three-part definition of playful math: (a) agency in exploration/goal accomplishment, (b) self-selection of mathematical goals, and (c) immersion, investment and/or enjoyment. The first and second authors coded each transcript separately and met weekly to reconcile discrepancies. All three authors then discussed any remaining questions or discrepancies. In addition to coding for playful math, we also analyzed the data for the two phases of activity identified in the literature, which we call *Explore-Focus Cycles* (Ellis et al., 2022). For Mathematician 1, we did this analysis as a whole group. For the remaining mathematicians, the first and second authors conducted their analyses independently, kept notes, and met weekly to discuss and reconcile codes. The project team met weekly to consider discrepancies and finalize codes until the codes had stabilized.

Findings

Playful Math Codes

We found evidence of two out of three playful math codes in all 13 mathematicians' problem-solving activity, and evidence of all three codes in nine of the 13 mathematicians' activity. The first code, *Agency in exploration/goal accomplishment*, occurs whenever a person has the agency within a task to explore their chosen ideas or work towards accomplishing their goals. The mathematicians almost always had agency to explore and accomplish their goals as a result of the open nature of the tasks, the structure of the interviews, and their authority as mathematicians. Even when the interviewer directed mathematicians' actions by asking them, for example, to prove a conjecture, the mathematicians typically retained agency to explore and select their own proof strategies. For example, while working on the Selfish Sets Task, Mathematician 5 (M5) discussed various strategies for answering the problem. She decided between "doing some answers and then prove it inductively" or "try to think about [the task] combinatorially in some way that would help the answer pop out." The interviewer acknowledged these different paths: "Yeah, so there are a lot of different ways that you could go from here" but did not suggest a specific direction. M5 immediately decided to "think about how I'm going to compute this sum". This demonstrated both *Agency in exploration,* as M5 thought through different approaches to solving the problem, as well as *Agency in goal accomplishment,* as she narrowed in on a direction to accomplish the task goal of solving the Selfish Sets Task.

The second code, *Self-selection of goals,* occurs when a person chooses their own goals or sub-goals for a task. These may include exploring the problem, making conjectures, deciding on a strategy, etc. For example, while solving the Interesting Numbers Task, M2 created a generalized algebraic expression for interesting numbers and wondered, "so now the question is what numbers can't be written in that form", which M2 began trying to answer using a parity argument. Similarly, in trying to determine how to prove that the number of selfish sets of $\{1, 2, \ldots, n\}$ is a Fibonacci number, M7 attempted to use a proof by induction on the size of the subsets: "Obviously you've got all the ones before, okay, so the proof would be induction. And, what does this new one add?" She tried to find a recursion but then switched to considering a sum of binomial coefficients. Before stopping the interview, she noted that in order to prove that the sum is a Fibonacci number using induction, she would need to "get a better sense" of the sum. Thus, her self-selected goals included using proof by induction, determining the recursive structure of the problem, and better understanding the sum of binomial coefficients.

The third code, *Immersion, investment, and/or enjoyment,* refers to when a person shows evidence of immersion or investment in the task, or shows that they are enjoying the task. For example, M1 demonstrated investment in the Selfish Sets Task by voluntarily returning to it. He spent 15 minutes on first task, shifted to the Selfish Sets Task for 20 minutes, and then returned to first task. After giving a proof for first task, he noted, "Wait, okay. So now I got a little time to play around with the other one. (Interviewer: Mm-hmm) Might not get it. Sets are not my thing particularly." He expressed an interest in returning to the Selfish Sets Task and attempting to solve it, which indicated investment. Additionally, during the interview, M1 expressed enjoyment with the two tasks, remarking, "These are great problems, by the way." Another indication of immersion or investment is losing track of time, which is one aspect of flow (Csíkszentmihályi, 1990; Liljedahl, 2018). We saw this when M4 was surprised at the time he had spent on the problem: “I've been working on this for 15 minutes, oh my god.” Other mathematicians persisted in solving the problem even when they were given a chance to stop, which indicated enjoyment and investment in the tasks. During M10’s interview, he continued working to solve the problem even when the interviewer provided an opportunity to stop:

Interviewer: Great. And just so you know, so it's been about an hour. So, if you want to stop now, that's fine. But if you want to try...

Mathematician 10: Oh, I'm just getting warmed up.

Interviewer: No please, please do. I just don't, I don't want to take up too much of your time.

Mathematician 10: Okay, no, no problem at all. Okay. Yeah, I should warn you, once I get going, I don't quit.

Thus, M10’s desire to continue working even being provided an opportunity to stop is evidence of investment in the task.

Recall that we found evidence of all three codes for only nine of the 13 mathematicians. For the remaining four mathematicians (M2, M6, M7, and M11), we saw evidence of *Agency in exploration/goal accomplishment* and *Self-selection of goals*, but we did not find evidence of *Immersion, investment, and/or enjoyment.* We found three reasons for the lack of immersion, investment, or enjoyment: (a) problem difficulty, (b) interruptions preventing sustained engagement, and (c) lack of open exploration. M2 was already familiar with the ideas in both problems, and so he was able to achieve solutions to both tasks in about 10 minutes. Thus, the problems were exercises for him rather than problems (Kantowski, 1981). M6 and M7 both experienced multiple interruptions during their interviews, which may have impeded their engagement and the ability to enter a state of flow. Finally, both M6 and M11 had a truncated exploration phase, as they found a viable solution path early on in their exploring. Without the extended exploration, these mathematicians may not have had as many opportunities to become immersed or experience enjoyment. In the next section, we report in more detail on the exploration phase by elaborating the evidence of Explore-Focus cycles in the mathematicians’ problem-solving activity.

Explore/Focus Cycles

We found that Explore-Focus Cycles (Figure 2) occurred in the tasks where mathematicians were playfully engaged. However, the amount of exploring actions versus focusing actions and the amount of time spent on those types of actions varied greatly from mathematician to mathematician. The two mathematicians for whom we found the most evidence of immersion/investment spent a significant amount of time exploring. Additionally, as we noted above, each of the four mathematicians who did not exhibit the immersion/investment code had fewer exploring actions and spent less time in exploration compared to the mathematicians who exhibited immersion, investment, and/or enjoyment. To exemplify the Explore-Focus Cycles, we present one cycle from M1’s engagement with the Interesting Numbers Task (Figure 2).

Within the cycle, mathematicians shift between exploring actions and focusing actions. *Exploring actions* occur when problem-solver(s) either (a) engage in an action with little or no anticipation of an associated outcome, or (b) notice properties of outcomes and wonder about the connection between those outcomes and the actions that led to them. For example, in the two tasks, mathematicians might generate small examples to look for a pattern or notice a general form to help them determine interesting numbers or minimal selfish sets. They may also explore different solution strategies before deciding to try one. A *focusing action* occurs when the action is tied to an anticipated outcome. For example, after generating small examples and noticing a pattern, a mathematician may make a conjecture, and then choose specific examples to test their conjecture. Both conjecturing and choosing examples are focusing actions.

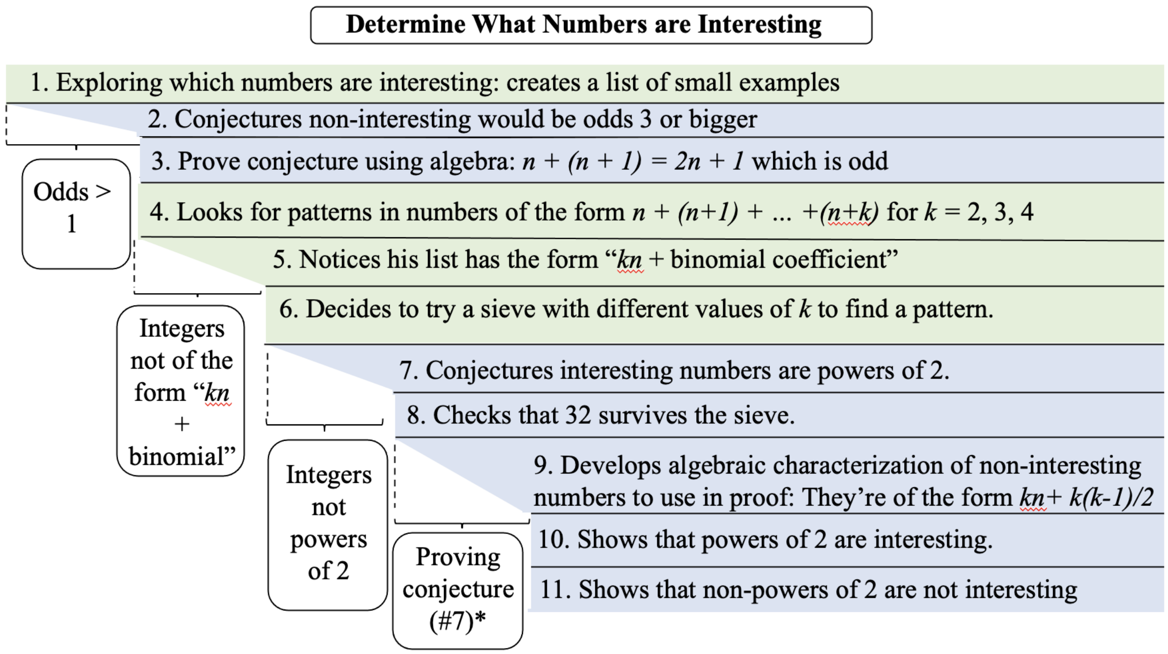

Figure 2. The Explore-Focus Cycle for M1's solution of the Interesting Numbers Task.

In Figure 2, the exploring actions are green and the focusing actions are blue. With our data, the problem space consisted of characterizing the objects specified by the tasks, namely, interesting numbers and minimal selfish sets. Once the mathematicians had a characterization in the form of a conjecture, they shifted to trying to prove their conjecture. The narrowing of the problem space can be seen in Figure 2 as "Odds >1", "integers not of the form kn + binomial coefficient", and "integers not powers of 2". Once M1 had conjectured that interesting numbers are powers of 2 (#7) and checked his conjecture with an example (#8), he shifted to trying to prove his conjecture, which resulted in another narrowing of the problem space. This action is denoted with an asterisk as this narrowing represents a shift from characterizing the interesting numbers to proving his conjecture about interesting numbers.

M1 began the task by exploring the constraints of the problem and generating small numbers (#1). His initial exploration only consisted of sums of two positive integers and led him to conjecture that the non-interesting numbers are odd numbers greater than 1 (#2). During his small example generation, he did not anticipate the form of non-interesting numbers; hence, his engagement was exploratory. He then made a conjecture after generating four examples. At this point, M1 shifted to trying to prove his conjecture, "non-interesting numbers are odd numbers greater than 1." He argued that if we consider two consecutive numbers, x and $x + 1$, then their sum is an odd number of the form $2x+1$ (#4).

At this point, the interviewer redirected M1 to the problem statement, pointing out that non-interesting numbers can be the sum of more than two consecutive integers. This shifted M1 back into a narrower problem space in which he tried to determine which even numbers are interesting (since he had already proved that odd numbers are non-interesting). To do this, he tried small algebraic examples (#4), i.e., he considered numbers of the form $3n + 3$, $4n + 6$, and $5n + 1$, which are the sums of three, four, and five consecutive integers, respectively. He noted that all of these integers are of the form $kn +$ binomial coefficient (#5). While generating these small examples, M1 had not anticipated the general form, nor at this point did he anticipate how it would be useful, thus he was still exploring. Deciding that he wanted to exclude all integers of those forms led him to comment, “this looks a little like the Sieve of Erastothenes … you’re trying to exclude numbers that have certain forms.” M1 then created a list of numbers and systematically eliminated them (#6); in doing so, he realized that the surviving integers are powers of 2 (#7). Although his decision to try a sieve focused his actions, he did not anticipate the outcome of the sieving process, and so we classify it as an exploring action.

At this point, M1 had a conjecture that the interesting numbers were powers of 2. In order to test his conjecture, he continued his sieving process to see if 32 survived the sieve (#8). This was a focusing action because he anticipated that 32 would survive. Having checked that his conjecture held for 32, M1 shifted to trying to prove his conjecture that a number is interesting if and only if it is a power of 2. At first, M1 did not see how to proceed and decided to spend some time with the other task. Once he returned to this task, he reviewed his work and noted that non-interesting numbers can be written in the form $kn + k(k\text{-}1)/2$ (#9). M1 first showed that if a number is a power of 2 then it doesn’t have the form $kn + k(k\text{-}1)/2$ since it would factor as $k/2(2n+k\text{-}1)$ which would be odd (#10). To prove the other direction, he showed that if a number is not a power of 2, then it is not interesting by demonstrating that every odd number can be written in the form $2n+k\text{-}1$ where k is a power of 2 (#11). All of his actions in the proof were focusing actions because he anticipated that they would allow him to reach a logical conclusion and prove his conjecture.

Discussion

Researchers have argued that mathematicians’ disciplinary engagement is playful (e.g., Mason, 2019; Su, 2020). Our study supports these assertions with empirical findings. We found evidence of all three aspects of playful math engagement (agency in exploration or goal accomplishment, self-selection of mathematical goals, and a state of immersion, investment, and/or enjoyment) for nine of our 13 mathematicians and evidence of the first two aspects for all 13 mathematicians. We also found evidence that the mathematicians shifted back and forth between the two types of engagement depicted in theoretical studies (e.g., Holton et al., 2001; Su, 2017), which we call exploring and focusing. Given the constraints present in analyzing task-based interviews, more research is needed to better understand the playful nature of mathematicians’ practice when engaged in their own research activities. Nevertheless, it is encouraging that we were able to verify theoretical claims with our findings.

What is perhaps more interesting is why some of the mathematicians did not demonstrate immersion, investment, or enjoyment. The first two aspects of playful math are ones that are seldom present for students, but we would expect to see in mathematicians. The third aspect, however, appears to be related to not only the ability to maintain engagement without interruption, but also to the level of exploration the mathematicians engaged in. Those whose exploration was truncated may not have found the tasks sufficiently challenging. This finding suggests that an appropriate level of task challenge, one that is within reach but is also not too easy, may be an important factor in fostering flow (e.g., Liljedahl, 2018), and, by extension, playful math engagement.

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