Explicating the Role of Abstraction During Analogical Concept Creation

Michael D. Hicks
Virginia Tech

Kyle Flanagan
Virginia Tech

Matt F. Park
Virginia Tech

Recent research has explored the potential for students to create new mathematical concepts by analogy with previously known concepts. However, there is more to be understood about the intricate processes of analogical reasoning that students leverage when creating new concepts. In this paper, we describe an initial theory explicating the role of abstraction during analogical concept creation. First, we introduce the abstracted domain to operationalize abstraction during analogical reasoning. We then describe two mechanisms of analogical concept creation through which abstraction may occur while reasoning about a ring-theoretic analogue to subgroups: (1) comparative analogy, and (2) inductive analogy. To broaden the interpretive scope of the theory, we further apply the mechanisms to scenarios involving meta-analogical reasoning and pedagogy. Implications for research and teaching mathematics with analogy are discussed.

Keywords: Abstract algebra, Abstraction, Analogical reasoning,

Analogical reasoning typically involves the comparison of two previously given or known concepts in search of relational similarity between the two (Gentner, 1983; Holyoak & Thagard, 1989). This forms the basis for many pedagogical analogies that are used in mathematics courses (Greer & Harel, 1998). For example, the concept of projection of a vector onto a general subspace is often presented with analogous references to spatial examples of a geometric vector projected onto a two-dimensional Euclidean plane, the goal being to ground students' knowledge of the abstract concept within a more accessible one.

While pedagogical analogies are often leveraged as a tool for reinforcing previously introduced concepts rather than creating new ones (e.g., Cobb, Yackel, & Wood, 1992), recent research has explored the potential for students to create new mathematical concepts, thereby expanding the scope of productive pedagogical analogies in undergraduate mathematics classrooms to allow for students to establish new mathematics for themselves. For example, it has been shown that students can explore, define, and create new concepts in ring theory by analogy with known concepts in group theory rather than simply reinforce their previous understandings (e.g., Hejný, 2002; Hicks, 2022; Stehlíková & Jirotková, 2002). This expanded view of pedagogical analogies in mathematics broadens the horizons for what pedagogical analogies can accomplish in advanced mathematics classrooms. However, to promote more productive pedagogical analogies in advanced mathematics, we must first better understand the intricate processes of analogical reasoning that students leverage when creating brand new (to them) concepts by analogy.

One avenue for deepening our understanding of analogical concept creation is to explore the role of abstraction within the process of analogical reasoning. Prior research in mathematics education has touched on the presence of abstraction in analogical reasoning (e.g., English & Sharry, 1996), although this research has not focused as heavily on cases in which students are generating new concepts or defining structures through analogy, focusing instead on developing conjectures to be verified (e.g., Lee & Sriraman, 2011). As such, the focus of this report is to develop an initial theory for explicating the role of abstraction during analogical concept creation. We are guided by the following overarching question: What is the role of abstraction during analogical concept creation?

Theoretical Underpinnings

The typical framework of analogical reasoning strives to capture very specific or 'perfect' representations of analogy rather than interpret potentially imperfect reasoning. In contrast, our proposed theory builds off the Analogical Reasoning in Mathematics (ARM) framework (Hicks, 2020) to interpret analogical activity that is specific to mathematics and may not align with usual or perfect cases of analogical reasoning. Analogies are viewed as series of mappings between a (typically) known source domain and a (typically) unknown target domain (Gentner, 1983), while analogical reasoning is the search for underlying structural relation between domains. A key component of the ARM framework is the adoption of the Actor-Oriented (AO) perspective (Lobato, 2012) to interpret analogical reasoning which allows for non-standard and idiosyncratic cases of analogical reasoning. During novel concept creation by analogy, immediate analogies are often impractical and instead require effort to refine the analogy. Thus, we draw upon the notion of *progressive alignment* (Gentner & Hoyos, 2017) to describe the process of developing an analogy through multiple analogical inquiries. In other words, we allow for the possibility for analogies to be incomplete at a given moment in time and consider them to be 'under construction' during analogical reasoning. Finally, for the purposes of this paper, we adopt Mason's (1989) notion of mathematical abstraction as a "delicate shift of attention" (p. 2), as well as acknowledging mathematical abstraction as a process closely linked to relational property recognition (e.g., Dreyfus, 1991; Ohlsson & Lehtinen, 1997; Piaget, 1985; Skemp, 1987). More specifically, Dienes (1967) indicated that "the essence of abstraction is to draw out common properties from different types of situations" (p. 201).

Developing the Theory: An Example of Student Thinking

To assist with presenting our proposed model of abstraction through analogical reasoning, we interlace the description of the general model with an example of analogical concept creation in abstract algebra. This particular example is based off of previously collected data of students' analogical reasoning in which the goal was to develop a ring-theoretic analogy to subgroups. Hence, the source concept is the subgroup concept, while the target concept is an analogous concept for rings.

To operationalize abstraction during analogical reasoning, we introduce the concept of the *abstracted domain*. We define the abstracted domain to be the domain to which the desired abstracted characteristics are mapped. Thus, the abstracted domain acts as a 'shell' of a concept or structure and forms the basis for future potential analogies. Figure 1 displays a possible abstracted domain for subgroups and subrings in which the notion of a 'sub-structure' would behave as a shell for future analogous concepts (such as subspaces of topological spaces.) The abstracted domain may vary in scope depending on the original concepts and domains being investigated.

The abstracted domain may not exist in every case of analogical reasoning. For example, a student may engage in a pure analogical comparison of two entities and construct a new concept without ever abstracting from the original structure by attending solely to surface level or superficial features. Instead, we claim that the abstracted domain exists only when abstraction activity arises during analogical reasoning. In alignment with ARM, we accept that the abstracted domain may be vague during its creation and may never be fully realized. Through progressive alignment, the abstracted domain may be refined over time as further inquiries are made. Our initial theory proposes two mechanisms through which the abstracted domain is developed: (1) comparative analogy and (2) inductive analogy.

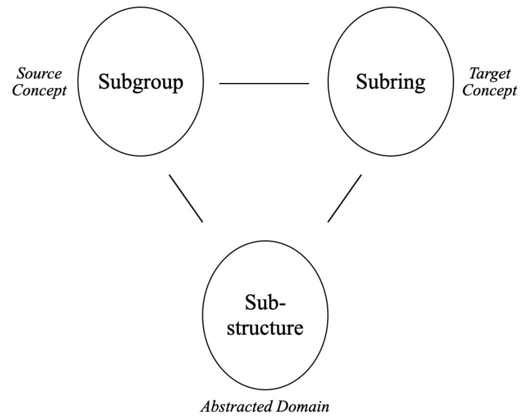

Figure 1. A possible abstracted domain associated with subgroup/subring.

Comparative Analogy

The first mechanism for developing the abstracted domain is *comparative analogy*. This process begins with a typical approach to analogical reasoning in which an analogy is either formed (or in the middle of being constructed) between the source and target domain. To illustrate this mechanism, we leverage an example of one student who had constructed an analogy to subgroups in the context of rings by directly mapping the subgroup structure and then making modifications to fit the ring context. After exploring the ring-theoretic analogy to subgroups, the student began to inspect his resulting analogy and reflect upon what he considered to be the key aspects of the two concepts of subgroup and the newly created concept of subring. During his reflection, some evidence of an abstracted concept appeared. In the following quote, the student is making observations about the subset similarity:

> *Student*: My idea is just a set and a subset. Like that, it lives inside of it.
> *Interviewer*: So you have this set and a subset idea? And that also carried over to this kind of context?
> *Student*: Yeah, so we would say this would imply S is, maybe let me draw a little ring [draws subset symbol with a small "o" over it]. Like, subring.

Thus, after having developed the concept of 'subring' and directly comparing to subgroups, this student began to recognize a broader concept of a 'set and a subset' that 'lives inside of it' that could potentially describe a larger class of analogous structures. The left figure within Figure 2 displays the process of comparative analogy in the context of comparing the subgroup concept and the new subring concept to establish the abstract concept of 'sub-structure'. The box in Figure 2 indicates that it was the analogy between the source and target that produced the abstracted domain.

Inductive Analogy

While comparative analogy requires a comparison of two concepts to create the abstracted domain, our second mechanism involves the development of the abstracted domain before the target concept is established. We refer to this second mechanism as *inductive analogy*. In contrast to comparative analogy, inductive analogy may occur with only one known concept to begin with (e.g., subgroup), as well as a possible awareness of other contexts (e.g., the subject of ring theory.) Knowledge of the original concept is then leveraged to develop the abstracted domain. A target concept is then created by making inferences about how the abstracted domain may be mapped onto the target. Consider the following quote by another student who was in the process of creating the subring concept by analogy:

> Yeah, well that's pretty much the definition for groups, if we have some group G and some operation *. Then if H is a subset of G and H is a group under that same operation, then we say H is a subgroup of G. **So, I literally just took the exact same thing from group theory and applied it to rings**.

We view the last sentence as partial evidence that this student had identified the abstracted concept of a sub-structure based on the subgroup concept.[1] Afterward, the new sub-structure concept is then mapped to the context of rings to form what the student eventually described as a 'subring'. The visual on the right of Figure 2 displays the mechanism of inductive analogy wherein the source concept of subgroup is leveraged to create the concept of 'sub-structure' before subring. We note that it is possible that multiple source domains are leveraged as the 'base case' for the abstracted domain. In those cases, progressive alignment may occur through multiple instances of comparative analogy preceding the induction and the creation of the new target structure.

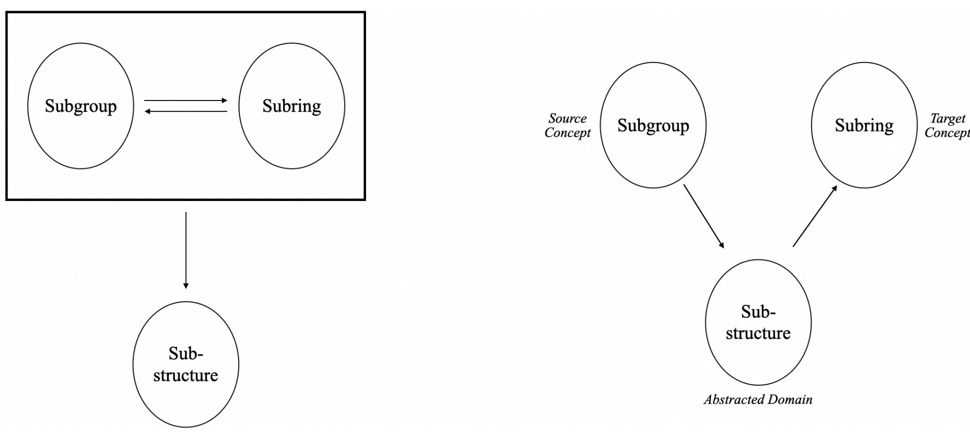

Figure 2. The mechanisms of comparative analogy (left) and inductive analogy (right).

In this section, we have introduced the mechanisms of comparative and inductive analogy to describe ways through which abstraction may occur during analogical concept creation. In the section that follows, we broaden the scope of this initial theory and apply it outside of the context of interpreting students' analogical reasoning between two structures in abstract algebra by looking more deeply at how this theory might inform purposeful analogical concept creation, and how this theory might be applied to pedagogy.

Further Applications of the Theory

Identifying Strategies for Purposeful Analogical Reasoning

In the absence of prior opportunities to reason by analogy in advanced mathematics, the students in the above examples may have been reasoning by analogy about advanced

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[1] We choose this example to clearly display reasoning with one domain rather than comparing across two domains. However, we recognize that the student may have been carrying the subgroup concept itself into the ring context without having created the abstracted sub-structure concept.

mathematics in a manner that was brand new to them. Overtime, learners may develop *meta-analogical awareness* (Modestou & Gagatsis, 2010) and, by extension, learn to purposefully and strategically reason by analogy. In particular, the increased awareness of one's own analogical reasoning can widen the range of strategies for analogical concept creation. For example, purposeful reasoning by analogy is a known strategy used by research mathematicians when defining new concepts (Ouvrier-Buffet, 2015). In this section, we describe how the mechanisms of comparative and inductive analogy clarify what one case of purposeful reasoning might entail when defining a new concept. To show this, we describe a hypothetical example of a student purposefully defining topological subspaces by analogy with several known algebraic concepts.

Consider a student who possesses an understanding of several algebraic structures and has recently learned of the definition of a topological space (i.e., a set X along with a collection of subsets containing X, the empty set, and is closed under arbitrary union and finite intersection.) Their experience in algebra suggests that given any particular algebraic structure, one can construct an additional one. Thus, upon becoming aware of the context of topology, an inductive analogy can be invoked to begin considering the meaning of a topological subspace. First, the student understands that subgroup, subring, subalgebra, or algebraic sub-structures in general are modeled by taking a subset of the collection of elements with respect to the structure in some way (i.e., the subset must be closed under the algebraic operation(s) of the over-structure.) In this way, the student may have already created an abstracted domain relevant to algebraic sub-structures. The student may then inductively analogize the notions of both 'subset' and 'respect the structure' from algebra within topology to establish the concept of a topological subspace. For this mathematically experienced student, it may be straightforward to argue that because an algebraic substructure is modeled with subsets, a topological substructure should be modeled by subsets as well. However, the student can also begin investigating what it means for a substructure to 'respect the structure' in the context of topology. In this way, purposeful inductive reasoning has provided an opportunity for rich exploration for this student to begin thinking about how a topological subspace may be defined.

After having established some notion of topological subspace (whether it is complete or not from the student's perspective), suppose now that the student encounters the definition of a topological subspace in a textbook or in a class lecture. The student may now reason by comparative analogy while also being informed by the findings from the prior inductive analogical reasoning. As an algebraic structure is a set along with various operations and a topological space is a set along with a collection of open subsets, the student may try to draw an analogy between operations and collection of open subsets. While the subset analogy was perhaps trivial, the student may not be immediately sure of the analogy of the restriction of the operations within a topological space. The modeling of topological spaces purely in the language of sets may motivate them to understand n-ary algebraic operations as sets as well. Particularly, an n-ary operation could be modeled as a subset of the $(n+1)$-fold cartesian product of the underlying set, and the restriction to a substructure can be realized by considering the restriction to the substructure to be the sequences that only contain its elements. Thus, they are drawing a comparative analogy between a subset of the cartesian product of a set with itself and a subset of the power set of a given set. In negotiating a satisfactory definition of a topological subspace, they are drawing on comparing different kinds of sets: they are attempting to understand the notion of topological subspaces by drawing comparisons between the cartesian product of set and the power set construction.

Applications to Pedagogy

Students may engage in analogical reasoning in any classroom setting. While the prior examples linked our proposed theory to interpreting student thinking and analogical reasoning, our final example links our proposed theory to teaching and curriculum design. In particular, we assert that increased attention to comparative or inductive analogy can assist teachers with identifying affordances or obstacles to promoting analogies in specific ways as well as consider the timing for productively introducing an analogy. In this section, we consider an example in undergraduate linear algebra. An important analogy that helps link many of the concepts in this course is the three-way analogy between systems of linear equations, matrices (or matrix-vector equations), and linear transformations (Larson & Zandieh, 2013). Every system of n linear equations in m variables can be represented with a matrix-vector product using an $n \times m$ coefficient matrix (or an $n \times (m + 1)$ augmented matrix), and vice versa. Similarly, every linear transformation T: $\boldsymbol{R}^m \to \boldsymbol{R}^n$ has a standard $n \times m$ matrix representation, and vice versa. This ability to transform between systems of equations, transformations, and matrices leads to many analogous theorems and properties in linear algebra, including the infamous Fundamental Theorem of Invertible Matrices.

One concept that is closely tied to this analogy is matrix multiplication, as matrix multiplication corresponds to compositions of linear transformations. Cook et al. (2018) conducted a textbook analysis on 24 introductory linear algebra textbooks, analyzing how matrix multiplication was presented in each text. They identified three overarching sequences for when and how matrix multiplication was introduced, and characterized these sequence rationales utilizing Harel's (1987) classification of linear algebra content sequencing. Pedagogically, the choice of which of these sequences is used for course design influences what analogical reasoning students engage in during class. Students' potential analogical reasoning in two of these sequences is provided. It is important to emphasize that neither sequence is considered better, but rather these examples demonstrate that pedagogical choice can influence students' analogical reasoning, and this framework provides a way to understand that reasoning.

Many of the common undergraduate linear algebra textbooks for large universities begin by introducing systems of linear equations and immediately transition into basic matrix operations, such as (typically in this order) addition, scalar multiplication, and multiplication (e.g., Larson, 2016; Poole, 2014). Textbooks with this structure often fall into the sequence where matrix multiplication is first defined using the matrix-matrix product AB determined by the dot products of row and column vectors, and the justification for the definition's rationale is postponed until much later in the course. Under this course design, after students have been provided with the definitions for matrix addition and scalar multiplication, it is possible that students may engage in inductive analogical reasoning, where these definitions induce the creation of an abstracted domain encompassing matrix operations as component-wise operations. Though this would lead to the 'incorrect' definition, it should be noted that this is reasonable analogical reasoning; hence, it should not be surprising that students are perplexed when the actual definition is presented.

Another common sequence for matrix multiplication includes first defining matrix multiplication with the matrix-vector product Ax as a linear combination of column vectors, which follows an introduction to systems of linear equations and linear transformations (e.g., Lay et al., 2015). Since the three-way analogy has already been introduced to the students, when some students see the full definition of the matrix-matrix product AB, they might construct a comparative analogy using any two (or all three) of those domains. In particular, some students may construct or refine an abstracted domain about 'how vectors are transformed.' In addition, prior to seeing this full matrix-matrix product definition, some students might engage in inductive analogy, hypothesizing what the actual definition is, being informed by the matrix-vector product definition. However, it is entirely possible that students will not engage in either of these forms of analogical reasoning.

Implications for Theory and Future Research

In this paper, we set out to describe an initial theory intended to explicate the role of abstraction during the process of analogical concept creation. To this end, we described two mechanisms through which abstraction may occur and illustrated several scenarios in which these mechanisms assist with interpreting or predicting possibilities of students' thinking. These mechanisms contribute to the literature on analogical reasoning in mathematics education, especially with respect to analogical concept creation. In particular, we greatly expanded the scope of possible scenarios presented by English and Sharry (1996) and Lee and Sriraman (2011) by both operationalizing and clarifying what abstraction during analogical reasoning might look like.

In addition to the examples presented in this paper, this theory can be applied to various other contexts, demonstrating even further potentially far-reaching implications. For instance, one can frame both problem-solving strategies (Weber & Leikin, 2016) and problem-solving analogies (English, 2004) in this theory. When an individual has solved two different problems with an analogous problem-solving strategy, there is a possibility that this individual will retroactively engage in comparative analogy, constructing an abstracted domain containing a 'problem-solving heuristic or strategy.' Moreover, an individual may also be in the process of solving a problem and is looking for ways to complete the problem. In this scenario, the individual could engage in inductive analogy, by reflecting on how a previous problem-solving technique could apply to the current problem, and constructing an abstracted domain for the general problem-solving strategy. This can also apply to proof-writing and proof-constructing strategies (e.g., Zazkis, Weber & Mejía-Ramos, 2015). Furthermore, researchers have demonstrated the value of example-based reasoning in mathematics (e.g., Lockwood et al., 2016; Lynch & Lockwood, 2019). It is possible that two or more examples give rise to comparative analogy, where the abstracted domain is an analogous property, concept, or new mathematical definition. Additionally, after an individual has meaningfully engaged with one example, and is about to engage with a subsequent example, this person may engage in inductive analogy. Prior to working out the details, the individual may construct an abstracted domain containing certain potentially relevant properties, which help the individual to predict how the example may work.

Though this paper has outlined the central components of this theory about the role of abstraction in analogical concept creation, and demonstrated its implications to various facets of mathematics education research, we have several future goals for expanding on and refining the theory. First of all, two mechanisms for this theory were introduced, comparative and inductive analogy; however, we acknowledge that these mechanisms do not always function entirely separate from one another. The example of the algebraically mature student engaging with topology explicates how one might engage in inductive analogy and then comparative analogy, and we would like to further delineate how these mechanisms interact. Second, we would like to more specifically explore how this theory is connected to theories of abstraction. In particular, we recognize that the construction of an abstracted domain aligns closely with Piaget's reflective abstraction (Beth & Piaget, 1966; Piaget, 1977, 2001). To what extent these two mechanisms entail reflecting and reflected abstractions versus empirical or pseudo-empirical abstractions is worth further exploration.

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