Physics Students' Interpretation of Matrix Multiplication

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Most upper-division physics courses use mathematics beyond that encountered in the calculus sequence. Matrix multiplication and eigentheory are used by physics students in courses including classical mechanics, optics, and quantum mechanics. While there have been recent investigations of student use of linear algebra in quantum mechanics, no prior work has examined how physics students approach this mathematics in other contexts. In this paper, we investigated the extent to which student interpretations of matrix multiplication could be characterized by those from Larson and Zandieh (2013): linear combination, system of equations, and transformations. We have examined student responses to written questions and used the results to develop an interview protocol to examine how physics students interpret matrix equations using the three interpretations to classify their responses. Results are shown from an interview sample of six undergraduate physics majors.

Keywords: Linear Algebra, Mathematical Reasoning, Physics

Introduction

It is well established that upper-division physics courses have increased mathematical difficulty and that students in these courses encounter a variety of mathematical topics beyond calculus, including linear algebra and ordinary and partial differential equations. From linear algebra, matrix multiplication and eigentheory are used most in physics. The most well-known application is in quantum mechanics, but matrices are also used in other courses including classical mechanics and optics. Prior work on student knowledge of linear algebra in physics contexts has primarily focused on quantum mechanics (e.g, Wawro et al., 2020), which brings into play a set of conceptual and representational issues that may or may not reflect student understanding of the underlying mathematics. For this project, we have chosen to look at student interpretation of matrix multiplication in a broader array of topics, including some acontextual mathematical tasks as well as contexts drawn from classical mechanics and optics.

Theoretical Framework

Several models have been proposed to describe student use of mathematics in physics. In each, successfully executing the mathematical procedure in question is only one element of success. Redish and Kuo (2015) proposed a framework for student usage of mathematics in science courses describing stages of modeling, processing, interpreting, and evaluating. For the specific case of upper-division physics courses, Wilcox et al. (2013) proposed the ACER framework 'to guide and structure investigations of students' difficulties with the sophisticated mathematical tools used in their physics classes.' In this framework, students activate the appropriate mathematical tool in addition to constructing a model, executing the math, and then reflecting on results. For earlier work, we were guided by the physical-mathematical model developed by Uhden et al. (2012). This model describes three skills used at the boundary of math and physics: mathematization (converting a model into mathematical expressions), technical operation (executing relevant mathematical procedures), and interpretation (translating a mathematical result into physical interpretations and implications). In a previous study, we characterized student written responses to a number of tasks, and coded responses for each. Technical operation was seldom the most challenging for students; rather, more than half of the students had difficulty in converting a physical model into a matrix equation (mathematization) and relating eigenvalues to system behavior (interpretation) (Her & Loverude 2020).

Given that interpretation of matrix multiplication is both a critical skill and a source of student difficulty, we focused the next portion of the study on how students interpret matrix multiplication. Larson and Zandieh (2013) described three mathematical interpretations of the matrix equation A**x**=**b** among students in linear algebra courses: Linear Combination (LC), System of Equations (SOE) and Transformations (T). In previous work, Larson (2010) described two mathematical procedures to matrix multiplication, 'Matrix Acting on a Vector' (MAOV) and 'Vector Acting on a Matrix' (VAOM) which can be connected to (T) and (LC), respectively. The three interpretations have distinct conceptual and procedural interpretations [see Table 1].

*Table 1. A brief description of each interpretation from Larson & Zandieh (2013) for the matrix equation, A**x** = **b**.*

Linear Combination (LC)	**System of Equations (SOE)**	**Transformations (T)**
• Vector **x** components act as weights that modify the column vectors of matrix A • Associated with 'Vector Acting on a Matrix' (VAOM)	• **x** is a solution that coordinates the values for **b** with matrix A as coefficient entries	• Matrix A operates on input vector **x** where vector **b** is the modified output vector • Associated with 'Matrix Acting on a Vector' (MAOV)

Their work in the context of mathematics courses led us to consider the usage of these interpretations among physics students applying matrix multiplication in upper-division physics. In informal conversations with physics instructors, most expressed a strong preference for a transformation perspective, perhaps due to the use of matrix multiplication in quantum mechanics. This guided the formation of the following research questions, the first of which is addressed in this preliminary report:

To what extent do physics students use each of the three interpretations from Larson & Zandieh, and do they express preference for any of the three?

What relationship, if any, does the preference for interpretations have to solutions to physics tasks using matrix multiplication and eigentheory?

Methodology and Data Analysis

This project involves qualitative research on student learning. Data collection involved individual interviews with undergraduate physics students ($N = 6$) at a public university serving a diverse student population. All students were physics majors; one was a double major (computer engineering) and one had switched from engineering to physics. Their math and physics backgrounds were similar with some variations; in particular, three of the six students had taken a linear algebra course. All of the students had previously taken a course titled 'mathematical methods in physics,' intended to provide support in more advanced mathematics for later upper-level physics core courses. The mathematical methods course included a brief (2-3 week) introduction to matrix multiplication and eigentheory. Four of six student had subsequently completed a quantum mechanics course in which matrix multiplication and eigentheory were used frequently. The interview sample included three men and three women; three were white, two Latinx, one Asian.

Due to COVID-19, our standard routine of conducting interviews in-person had to be modified for an online setting. The first three interviews were recorded and conducted through Zoom, and the remaining three were in person. The interview protocol was designed to collect information on how physics students think about matrix multiplication and analyze the bridge between their mathematical interpretations and their physical interpretations. The interview consisted of nine questions that were separated into three sets. The first set were acontextual tasks intended to explore how students interpreted matrix multiplication without any physical context [see Figure 1] and were influenced by prior RUME studies (Henderson et al., 2010). The second set focused on reflection and rotation matrices along with the eigenequation. The last set of questions included three different physics concepts that used matrix multiplication and the eigenequation (optics, coupled blocks on springs, and a quantum spin operator), though some students did not complete all of these tasks. The first set of tasks was used to probe which of the interpretations, if any, were articulated by students Questions 1 and 2 provided examples of matrix multiplication and a matrix equation and asked students for their interpretation; question 3 provided three expressions associated with (LC), (T), and (SOE), respectively, and asked students to describe how, if at all, each expression represented the matrix equation.

Portions of the interview transcripts were coded iteratively by two members of the research team. Interview coding was in part emergent and in part driven by prior research. The primary focus of our analysis procedure was to investigate the students' interpretation

Question 1	**Question 2**	**Question 3**
Consider the expression $\begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. How do you think about this expression?	Consider the matrix equation $\begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. First, how do you interpret this?	$\begin{pmatrix} 1 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ For each of the following expressions, explain how if at all, it represents the matrix equation. (i) $x\begin{pmatrix} 1 \\ 8 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \cdot x + 2 \cdot y \\ 8 \cdot x + 4 \cdot y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ (iii) $x + 2y = a$, $8x + 4y = b$

Figure 1. The first set of interview tasks.

rather than assessing 'correctness'. The initial analysis was based on the interpretations described by Larson and Zandieh (2013); student utterances were coded as (LC), (T), or (SOE). The grammar of student utterances was part of the analysis; e.g., transformation interpretations were often characterized by transitive verbs like 'manipulated,' 'changed,' or 'rotated.' MAOV and VAOM were coded and associated with (T) and (LC), respectively. The transcripts were also examined for evidence of interpretations that were not accounted for by any of these codes.

Initial analysis suggested that additional refinements would be required. Some student responses seemed to shift depending on whether they were primarily procedural or conceptual. This led to creating a parallel set of codes in which responses were identified as being procedural (relating to the method used to evaluate the mathematical expression, e.g., 'the first row with the column of the second matrix'), conceptual (how they make sense or meaning of the expression or outcome, e.g., 'change a vector from one initial state to a final state'), graphical, or none of these. An utterance might be coded as, say, both procedural and LC. These distinctions mirrored analysis in Larson and Zandieh (2013) as well as other recent work (Serbin et al., 2020). The researchers worked to resolve differences in coding and adjusted the codes appropriately; one example of this process is described in the Results section. The analysis process consisted of grouping terms and sections of the interview transcripts and examining the resulting data set for trends and patterns. Responses were further examined for either, on one hand, consistency of interpretation or, on the other, fluid, changing interpretations across different tasks.

Results

Table 2 presents a summary of our analysis. Student descriptions of matrix multiplication were generally consistent with the three descriptions (T), (LC) and (SOE). Nearly all student statements were readily aligned to one of the three.

There was some degree of preference for the Transformation interpretation (T), with some variation. All six students incorporated the (T) interpretation at some point. Three of the six students consistently interpreted matrix multiplication, in a variety of tasks, in terms of the transformation interpretation and the associated MAOV description, with few or no codes for (LC) or (SOE). This was a clear signal throughout these three interviews, and was often accompanied by explicit rejection of, or confusion with, equations in Question 3 associated with other interpretations. Three of the six students showed signs of more fluid interpretations, with all articulating (LC) and/or (SOE) interpretations, though again all shifted to (T) at some point during the interview. Most of the students offered both procedural and conceptual support for their preferred explanations, but one of the six remained focused on procedural knowedge and offered little in the way of conceptual interpretation.

Below we present extended examples of responses given by students in different categories: student A, a consistent user of T interpretations, and student C, whose responses were coded as supporting all three of the interpretations at various parts of the interview.

Table 2. A summary of student backgrounds and responses. LA/QM refers to whether students had completed linear algebra and quantum mechanics, respectively. Conc/Proc = conceptual and procedural.

Student	A	B	C	D	E	F
Major(s)	Physics	Physics	Physics (formerEng)	Physics	Physics +CompEng	Physics
LA / QM	Y/Y	Y/Y	Y/N	Y/Y	Y/Y	Y/N
Initial / Final Interpretation	T / T	T / T	LC / T	T / T	SOE / T	LC / T
Stable/Fluid	Stable	Stable	Fluid	Stable	Fluid	Fluid
Conc/Proc	Both	Proc	Both	Both	Both	Both

Student A

Student A was coded as consistently using a (T) interpretation throughout the interview. For the first two questions, Student A immediately described characteristics (T) interpretations, e.g., using terms 'manipulated' and 'transformation matrix': "So uh I see this as we got some of vector quantity here and it's being manipulated by some sort of transformation matrix."

Student A's responses further suggested a strong association of the (T) interpretation with the procedure of matrix multiplication. On question 3 he immediately chose the second equation and commented that choices (i) and (iii) were difficult to parse conceptually. He was unsure whether these were even mathematically equivalent, stating that (iii) was 'divorced from the whole process of matrix multiplication.' After some discussion, student A performed procedures to reproduce the expressions, and once he was satisfied that all three were representations of the matrix equation, he reversed course somewhat and stated that choice (i), the (LC) interpretation, was the best representation. However, his conceptual description in support of choice (i) was coded as a (T) interpretation as he referred to 'transforming' the elements of the vector and 'what the matrix is doing,' rather than the weighting of column vectors associated with (LC). Indeed, he interpreted all three expressions in terms of (T), though the intent was that each expression mapped to one of the three interpretations.

Student A: "…number (i) seems like it is more representative…it represents what the matrix is doing more than the other two. So the fact that this [points to first column of matrix A] is transforming that [points at x] and this [points to second column of matrix A] is transforming that [points at y] kind of represents the idea of the matrix multiplication a bit more than—a lot more than this [option (ii)] and more than this [option (iii)] I would say. Because we see a component of the matrix acting on sort of a scalar."

Student C

Coding student C's responses could be challenging, in part because she used unconventional terminology. For example, in her response to question 1, she explained that while the knew the accepted terminology, she preferred to refer a 2 by 2 matrix as a "2 by 4." She gave responses consistent with each of the three interpretations at different moments in the interview, offering both procedural and conceptual explanations. In her initial procedural response, she showed multiplying the rows of the matrix with the column vector, consistent with MAOV. However, her subsequent conceptual description was coded as being consistent with VAOM and thus (LC). In question 2, she described the matrix equation as: "$\begin{pmatrix} x \\ y \end{pmatrix}$ is modifying $\begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}$ where $\begin{pmatrix} x \\ y \end{pmatrix}$ would be some arbitrary modification being performed to get an arbitrary outcome $\begin{pmatrix} a \\ b \end{pmatrix}$." The vector 'modifying' the matrix suggests VAOM, but written work was consistent with MAOV.

Ultimately, Student C concluded that choices (i) and (iii) were possible representations of the matrix equation but was ambivalent because she could not fully track their procedures. She identified choice (ii) as 'no representation' and rejected its validity altogether.

Conclusion and Implications for Future Work

Analysis reveals some patterns that may have implications for instruction as well as future research efforts. Three of the six subjects showed a strong and consistent preference for the (T) interpretation. While each of the choices in task 3 was intended to support one interpretation, these students provided (T) interpretations for all three. The other students all used (T) at some point but offered more fluid responses, shifting among the interpretations. The preference for (T) interpretations may reflect a difference in disciplinary usage, as recent experiences with matrix multiplication came in physics courses that likely emphasized the (T) interpretation above others.

While the sample is small, two additional patterns are of note. First, the three students who had the most consistent (T) interpretations had all completed two semesters of quantum mechanics, and two of the three who shifted among interpretations had not yet completed the course. Quantum mechanics instructors and texts frequently use language like 'the operator acts on the vector', which seems to be consistent with the (T) interpretation. Second, the two students with engineering backgrounds were both among those who used other interpretations; this may suggest additional disciplinary influences.

Analysis is ongoing and later publications will include analysis of later parts of the interviews including physical contexts, as well as written problems developed based on these interview tasks. Informal discussions with instructors for relevant physics courses suggest a marked preference among at least these physics instructors for the (T) interpretation, possibly suggesting a further avenue of inquiry.

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