A Student's Application of Function Composition when Solving Unfamiliar Problems: The Case of Zander

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Function composition is highly useful for making sense of many fundamental ideas in calculus and related areas of mathematics. However, of the large body of research investigating students' understandings of function, very little has been centrally focused on function composition. In an effort to contribute to the existing findings pertaining to function composition, this interview-based study involved an exploration of a first-semester calculus student's use of function composition when solving unfamiliar problems. The student's responses revealed potential evidence of his understanding of function composition, function as an action or process, and covariational reasoning.

Keywords: Function Composition, Function, Covariational Reasoning

Introduction and Literature Review

Functions serve a fundamental role for modeling dependence relationships between variables in every area of mathematics and physical science. Thus, it is not surprising that the idea of function is heavily emphasized in mathematics curricula, and has been heavily studied in mathematics education research (early examples include Breidenbach et al., 1992; Carlson, 1998; Vinner & Dreyfus, 1989). Function composition has also been mentioned as an area of importance, but often briefly or in passing. To date, few studies in mathematics education related to the idea of function have had function composition as their central focus (a few exceptions include Bowling, 2014; Engelke et al., 2005; and Kimani, 2008). However, applications of function composition can be found in many important ideas studied in calculus, including the limit definition of derivative, related rates problems (Engelke, 2007), and the chain rule (Clark et al., 1997; Cotrill, 1999; Engelke, 2007). Thus, function composition is an area worth studying further, particularly in the context of students' experiences prior to and during calculus.

Many of the existing studies pertaining to function composition have been based on the primary theoretical frameworks used to explain students' understandings of function more generally. Among the most common are Breidenbach et al.'s (1992) characterization of students' perspective of function in terms of specific actions or procedures to compute a result, or an "action" view of function, versus a general process that associates input with output values, or a "process" view of function (empirical studies using this framework have included Ayers et al., 1988; Bowling, 2014; and Engelke et al., 2005). Other frameworks used in function composition studies have included Carlson et al.'s (2002) framework for covariational reasoning (Bowling, 2014), a framework to study the impact of representational context (e.g., symbolic, tabular, graphical, word problem) on a student's conception of functions in a problem (Kimani, 2008), and Sfard's (1992) framework for function in terms of operations or in-progress activities, referred to as an "operational" view of function, or in terms of a single object that could be operated upon, referred to as a "structural" view of function (Kimani, 2008).

Several findings from existing studies pertaining to function composition have been focused on students' reasoning relative to the above frameworks. For example, studies employing the action and process framework have found that a process view of function can play an important role in student success with unfamiliar function composition problems, which the students would not be able to rely on a memorized procedure to solve (Ayers et al., 1988; Bowling, 2014; Engelke et al., 2005). Across multiple studies, another common finding is that students are often more comfortable solving function composition problems with functions defined symbolically than problems with graphical, tabular, or applied (word problem) contexts (Engelke et al., 2005; Kimani, 2008), which could be linked to the symbolic substitution approach to function composition commonly presented in high school curricula (e.g., Bowling, 2014). These studies have provided insight into how students' understandings of function might inform their approaches to solving function composition problems.

In addition to the above findings based on existing frameworks, a few studies have resulted in findings related to students' reasoning specific to function composition. One such finding was that, when solving related rates problems involving variables which the students found difficult to relate directly, students found a "middle variable" to be helpful for identifying familiar functions that could be composed to relate the primary variables of interest (Engelke, 2007). This finding suggests a possible extension of existing function frameworks to account for reasoning involved in function composition. In an effort to continue extending existing frameworks, this study was intended to address the question: In what ways does a calculus student apply their understanding of function composition, variable, and function to solve unfamiliar problems?

Theoretical Perspective

The analysis for this study involved a characterization of the types of functions and relationships between values a student might construct. For this work, two main frameworks for explaining students' understanding of function were leveraged: action and process view of function (Breidenbach et al., 1992) and covariational reasoning (Carlson et al., 2002; Castillo-Garsow et al., 2013). Each of these frameworks are described in more detail below.

Action and Process View of Function

Someone using an action view of function perceives a function as a specific set of computations or other describable, repeatable activities to get a particular result from given information. With an action view, a function is the method by which a result can be related to the given information (often both the result and given information are numerical values). For instance, someone with an action view of function might associate $f(x) = 3x + 2$ with the series of computations: 'multiply x by 3 then add 2'. On the other hand, someone using a process view of function perceives a function as an association between values which could be specified in different ways and is not tied to a specific rule. With a process view of function, someone might associate $f(x) = 3x + 2$ with an assignment of every real number (each of which would be a value of x) to a number that is 2 units greater than the number 3 times its value. With a process view, someone might imagine the rule of assignment according to the equation above, a graph, a computer program, a verbal description, an applied context, or a variety of other ways.

Covariational Reasoning

A commonly adopted perspective of covariational reasoning for modeling students' understandings of function is that of Carlson et al. (2002). Re-framed to account for the idea of function as a dependence relationship, Carlson et al. (2002) defined covariational reasoning as imagining a situation with multiple variables that change together such that the value of one variable depends on the value of the other. Underlying the idea of variable according to this perspective is an idea of quantity as conceiving of a feature of an object which one can imagine measuring (e.g., Thompson, 2011). The situation in which one might imagine a quantity could be a tangible, real-world situation, or it could be more abstract. For example, someone could quantify their distance from a wall in a number of feet, or they could quantify the x-coordinate of a point in the Cartesian plane as the horizontal distance of that point from the origin, measured in some specified unit used to scale the graph.

Carlson et al. (2002) characterized students' covariational reasoning into five stages of mental activity (MA1 – MA5). A description of the first three (used in this study), framed in terms of constructing a dependence relationship between variables is as follows: (a) MA1: identifying two varying quantities and noting that the value of one depends on the value of the other, (b) MA2: a qualitative description of the direction of change in one variable with respect to the value of the other; and (c) MA3: a description of an amount of change in one variable with respect to the value of the other. Castillo-Garsow et al. (2013) proposed two additional forms of covariational reasoning: chunky continuous covariation, through which someone imagines the variables' values changing in completed chunks or intervals, and smooth continuous covariation, through which someone imagines the variables' values changing gradually, smoothly, or through each moment in time. In addition to the forms of covariation above, in some situations, someone might conceive of a fixed quantity, or, in an algebraic context, associate a symbol with a fixed value (which they may or may not imagine as a quantity).

Methods

Data Collection and Task Design

The researcher (author of this paper) conducted a semi-structured interview with a first-semester calculus student, Zander, attending a large, public university in the Southwestern United States. Zander's calculus course involved a research-based curriculum, which could have impacted his understanding of function, covariation, and function composition differently from traditional calculus curricula. While solving the interview tasks, Zander was asked to think aloud. As he described his thinking and solution approaches, the researcher sometimes asked follow-up questions to further inform hypotheses about his thinking. Scanned images of any written work from the interview were also collected for further evidence of Zander's thinking.

The interview tasks were designed to be unfamiliar to most students, so that a solution would probably not be immediately available from experiences with similar problems. The researcher anticipated that a student's reasoning about the task context and solution attempts would give rise to some form of function composition, while also providing opportunities to generate hypotheses about their understanding of function and types of covariational reasoning.

Data Analysis

The author intended to leverage existing frameworks (as described in the theoretical perspective) and generate new explanatory models for students' applications of function composition in solving the tasks. In alignment with this goal, the author analyzed the interview using a process similar to that described by Simon (2019). Analysis consisted of three primary phases: (a) generating hypotheses about a student's thinking at specific moments in the interview; (b) generating hypotheses about a student's thinking over the course of the interview; and (c) modeling the student's understanding of function in the context of the interview.

Results

To illustrate the multiple types of function composition Zander appeared to use during the interview, data will be reported from two tasks in contrasting representational contexts.

The Equation Task

Let x be a variable that takes on real numbers, and let f be a function defined by $f(x) = x^2$. Suppose that y is a variable related to x through the relationship $y = 5x$. If y increases from -20 to 20, what will happen to the value of $f(x)$?

Figure 1. The Equation Task Prompt

The equation task was designed with the hypothesis that most students will imagine the equation $y = 5x$ as a way to directly relate a value of x to a value of y and the equation $f(x) = x^2$ as a way to directly relate a value of x to a value of $f(x)$, though the nature of the relationships students construct could vary widely. The prompt provided numerical information with which to associate y and requested that the student state a conclusion about $f(x)$. No equation or other function definition was provided to relate a value of $f(x)$ with a value of y, posing a potential challenge for many students in finding a relationship between y and $f(x)$.

Zander's Response to the Equation Task

Zander first responded to the equation task by stating an equation of the form $x =$ [expression containing y], but shortly afterward, indicated that the equation ultimately would not matter and gave a more general description of how he imagined the value of $f(x)$ might change with the value of y.

Zander: so x is equal to y divided by five I think, and—but y is from minus twenty (inaudible) but it does not really matter at the end because, uh, x squared, because it's, uh, x squared, so I think the value of y, I mean the value of f x will increase, when you go from minus twenty to twenty for y.

Zander's initial response suggested some details about his goal for solving the problem he appeared to perceive in the task and the way in which he might describe two variables he imagined, y and $f(x)$, changing simultaneously. However, there was almost no evidence as to why Zander had initially stated that "x is equal to y divided by five". Upon probing for further evidence, Zander's response revealed that writing an expression for x in terms of y was most likely part of a strategy that would allow him to compute—or imagine computing—a specific value of $f(x)$ from a specific value of y.

Zander: So, like, the…I think (inaudible) a function we're supposed to plug in the value of x, and then we're supposed to get another answer which is like our value of y or it could be something else too. So, in this we're like given two things. First of all, it's a function which is f x equals to x squared, so if we have a value of x, we just plug it in for, I'm talking about f x equals to x squared, and then we'll get our y, right? So if we plug in one, it'd be one. Plug in two, it'd be four. So, but (inaudible) to figure out what our y will be, uh, I mean our x will be, we are given another equation, which is that y equal to five x. So, I think from this equation, y is equal to five x, we're supposed to figure out what our x will be, and then we can plug in that x into f x equals to x squared and then we can figure out uh, another, like the, another function (inaudible) y for that one.

From Zander's statement at the beginning of the above excerpt, he appeared to be imagining a situation in which he would consider a specific value of x, perform a calculation, and obtain a specific value of y resulting from that calculation. It is not completely clear from Zander's overall response how he imagined the relationship between x and y to yield the equation $x = y/5$; however, given what appears to be a computational use for the equation with specific values of x and y, it is plausible that Zander imagined manipulating the equation using a "solving" or "switching" procedure. More evidence would be needed to confirm this. Zander also mentioned an overarching strategy for figuring out a value of $f(x)$ from a value of y that involved two separate calculations: a calculation that would give him a value of $f(x)$ from a value of x, and a calculation that would give him a value of x from a value of y.

The Cylinder Task

A chemist pours a liquid into two empty, differently-sized cylinders. The smaller cylinder has a radius of 2 cm, and the larger cylinder has a radius of 3 cm.

(a) If the chemist pours the same volume of liquid in both cylinders, and the height of liquid in the smaller cylinder is 11 cm, what is the height of liquid in the larger cylinder?

(b) If the chemist pours the same volume of liquid in both cylinders, as the height of liquid in the smaller cylinder increases from 0 to 11 cm, how does the height of liquid in the larger cylinder change?

Figure 2. The Cylinder Task Prompt

The cylinder task is loosely based on a problem from the Precalculus: Pathways to Calculus workbook (Carlson et al., 2020). This task was designed in anticipation that if a student wanted to relate the height of liquid in the smaller cylinder and the larger cylinder numerically, they would need some intermediate value (which they could imagine as fixed or varying) they could individually relate to both heights. Then, they could use the intermediate value to compose the relationships, forming a relationship between the height of liquid in each cylinder. One anticipated intermediate value would be the volume of water in each cylinder, since the task context specifies that the volume would be fixed, and it might occur to many students to leverage the formula for the volume in a cylinder with respect to radius of the base and height: $V = \pi r^2 h$ (this formula was not provided in the task context, but was given to a student upon request).

Zander's Response to the Cylinder Task

Zander appeared to engage in two primary forms of function composition in his response to the cylinder task, both of which involved a relationship between the volume of liquid in each cylinder and the height of liquid in each cylinder. The first composition involved computing a single value for the volume of liquid in the smaller cylinder, substituting the result into the formula for volume in a cylinder with respect to radius of the base and height of liquid in the cylinder, and solving for the height of liquid in the larger cylinder. Zander's computation of a single value for the volume in each cylinder and solving for the corresponding height in the larger cylinder enabled him to achieve the goal of getting a numerical answer. The type of function composition he appeared to use in this instance could be regarded as grounded in an action view of function (Breidenbach et al., 1992) and a fixed-value understanding of the volume and height of liquid in each cylinder. However, his description of the relationship between the height in each cylinder shortly before this series of calculations suggests that he imagined the height and volume as variables with which he could make a general, qualitative comparison. In fact, his first inclination for finding the height of liquid in the larger cylinder was to use the relationship between the radii of the cylinders to form a mathematical analogy with which to compare the heights.

Zander: Um so first of all, I'd say that the height of the liquid in the larger cylinder is going to be less than what it is for the smaller one, because the radius is less, which means that for the first one, it is like kind of small, so it will fill up more. And the second one is a larger cylinder, it has a radius of three, so basically its base is bigger, so it could fill in the same volume but in like less height, that is what I'm trying to say here.

Following his description, Zander tried to use a proportion to express the comparison mathematically and ultimately calculate a value for the height in the larger cylinder, but he was not confident that the equation he wrote would result in the correct calculation.

Zander: I'd say, when our radius was two, our height was eleven, and then that'd be, is equals to when our radius is three, what is going to be our height. I'm not really sure if this is like in a ratio or not, but it should be because after all it's like a cylinder, it's just different size. The volume is going to be the same, the amount of liquid poured is the same so, this should be like in a ratio with each other.

When asked why he decided to set up a proportion, Zander's response suggested he was trying to guess what equation he could set up to calculate the desired value: the height in the larger cylinder. He noted that there was little information given in the problem overall that he could use to achieve this goal. Thus, the interviewer asked him what other information would be useful for him to know, and after only a short pause, he stated that it would be useful to know the formula for the volume in the cylinder.

Interviewer: Um, what information would be useful for you to have, that would allow you to figure out the height of water in the larger cylinder?

Zander: (inaudible) Radius. Uh, I think for this, we need to know the formula for the volume of the cylinder. If we do know the formula of the volume, I don't remember it that well, maybe it is—I don't know, I don't remember.

Once the interviewer gave him the formula for the volume in the cylinder, he made a statement that suggested he imagined the formula as a way to express a dependence relationship between the enclosed volume, height, and radius of a cylinder.

Zander: Yeah, so our pi is always constant, so if you're making comparison, it does not really matter. We are only dependent on two things for the volume. The first one is our radius, and the height.

Following his use of the volume formula to calculate the height of liquid in the larger cylinder, Zander rejected the proportion he initially had set up, concluding that it would not be possible to have a greater height of liquid in the larger cylinder than in the smaller cylinder (he had calculated a greater height from his initial proportion, whereas the volume formula produced a smaller value for the height in the larger cylinder).

The second form of function composition Zander used emerged when prompted to describe how the height of liquid would vary in the larger cylinder. Zander described a comparison of two graphs he imagined: one for the height of liquid in the smaller cylinder with respect to volume, and the other for the height of liquid in the larger cylinder with respect to volume.

Zander: Our x is going to be the volume and our y is going to be the height…so if the volume is increasing for the smaller one, our height is also going to, height is going to increase in both of these, so we do know that the graph is going to be like linear in this, it'd be increasing, but at a different rate for the smaller one, it would increase at like a really high rate so our slope is going to be really high; however for the larger one it's not.

Following his description of the graphs and a discussion about how he conceptualized the "rates" he was comparing, Zander described what first appeared to be evidence of a smooth continuous conception (Castillo-Garsow et al., 2013) of the covariation of each height with respect to volume. After further discussion, the researcher concluded that Zander was most likely employing a chunky continuous conception of covariation, since he stated that the slope of the graph should be the same over any size of interval for the volume.

Zander: So like we can, in this equation, we can find out our, like, height at every single moment as volume changes, so as of here, our volume was one thirty eight point one six, and we figured out that our height would be four point eight, so we can increase that volume to one forty and then we can figure out our height. This is what we can do. Then we could like have a slope at this point, and I hope that this slope is the same for the, every single interval in there, otherwise it won't be linear.

Conclusion

In response to both tasks, Zander frequently alternated between computing specific values and using algebraic manipulation procedures, and making general descriptive comparisons between what he seemed to imagine as variables. Zander appeared to imagine the functions as a means of relating the direction of change in two variables (Carlson et al., 2002). Thus, it appears that Zander leveraged a process view of function (Breidenbach et al., 1992). However, whether he used an action or process view of function in particular parts of his solution depended on his goal at that point in his thinking process. When his goal was to calculate a specific value or define an algebraic equation, he described a series of computations between individual values, suggesting he employed an action view of function in these situations.

In addition, Zander appeared to engage in several types of function composition. One form of function composition involved a series of computations and substitutions to compute one specific value given another (i.e., computing the height of liquid in the larger cylinder from a known height of liquid in the smaller cylinder). A second form of function composition involved algebraic manipulations to link together two equations, followed by a two-step sequence of computations to relate several numerical values, one-at-a-time, input to the first equation to corresponding numerical values output by the second equation (i.e., using the equation $x = y/5$ and subsequently the equation $f(x) = x^2$ to compute values of $f(x)$ from values of y). Finally, a third form of function composition involved a sequence of covariational relationships through which he appeared to imagine a chunky continuous, or possibly even smooth continuous variable (Castillo-Garsow et al., 2013) in relation to another by comparing his visualization of two graphs (i.e. comparing volume-height graphs he constructed in response to the cylinder problem).

Zander provides an example of how one student can use several different forms of function composition, covariation, and function to solve unfamiliar problems. His case suggests that the type of function composition that might be deemed applicable in a problem situation depends on the experiences with covariation and function someone associates with a problem's representational context and the goal someone has for a solution at particular points in the solution process.

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