Student Reasoning about Definite Integrals on a Calculus Concepts Assessment

Toni Hall
Oklahoma State University

Beth Cory
Sam Houston State University

Taylor Martin
Sam Houston State University

Vicki Sealey
West Virginia University

Michael Oehrtman
Oklahoma State University

Assessing student understanding of calculus concepts is of interest to RUME researchers and practitioners alike. This paper details the initiation of the development phase of a new instrument to assess student understanding of the central concepts in the introductory collegiate calculus course. We discuss a framework for defining "conceptual understanding" and make transparent our process for selecting and refining robust tasks to measure it. We provide discussion on interview responses that reveal how students may attempt to rely on memorized procedure and, as a result, discredit productive conclusions. Finally, we outline and seek discussion on future work to continue development of valid and reliable questions to assess the broad range of productive understandings in calculus.

Keywords: calculus, conceptual understanding, assessment

Introduction

Although notable strides have been made by researchers such as Epstein (2007) who created the Calculus Concept Instrument (CCI), no curriculum-independent concept inventory for Calculus I has been validated and found to conform to accepted standards of educational testing (Gleason, Thomas, Bagley, Rice, White, & Clements, 2015; American Educational Research Association, 2014). The purpose of our study is to initiate the development of a valid and reliable instrument to assess students' understanding of central concepts in an introductory college calculus course for Science, Technology, Engineering, and Mathematics (STEM) majors. Our goal is to make our process and criteria transparent so that (a) researchers and educators can make informed choices about what portions they want to include in their research and (b) others in the community can contribute to the instrument in a modular way. Our initial research focuses on interpretations of accumulation and area for definite integrals.

Literature Review and Theoretical Framing

Graphical representations of definite integrals as signed area under a graph are ubiquitous in introductory calculus courses. Research on students' understanding of integrals shows that while area interpretations are widely adopted by students, they are often misapplied and hinder further productive quantitative reasoning (Jones, 2015; Sealey, 2006). Past research has also shown that an adding-up-pieces conception of a definite integral is critical for successfully modeling and interpretation especially in STEM applications (Meredith & Marrongelle, 2008; Von Korff & Rebello, 2012; Jones, 2015; Oehrtman & Chhetri, 2015). Sealey (2014) documented the nature of such student understanding paralleling the mathematical structures of products, sums, and limits in the formal definition, and Jones (2015) labeled the application of such reasoning as a Multiplicatively-Based Summation (MBS). Simmons and Oehrtman (2017) extended this analysis to Quantitatively-Based Summation (QBS) in which the basic models used to construct the parts to be added in the definite integral are not necessarily conceived as a product of the integrand and change in input.

Tallman, Reed, Oehrtman, & Carlson (2021) found that the productive meanings assessed on calculus exams at colleges and universities in the U.S. are sparse. The concept of a definite integral as area is not always supported by quantitative reasoning. Students often approach integration as simply finding the area of a planar region without considering an underlying meaning of accumulating a quantity. Tallman et al. (2021) identified key features of assessment tasks that improve their potential to assess students' productive understandings including that the items should require students to (a) coordinate interpretations, (b) provide explanations or justifications, (c) make comparisons, or (d) draw inferences. Such features require students to make strategic decisions in the problem-solving process rather than only select and apply a memorized procedure.

Methods

After reviewing the literature on student understanding and learning of definite integrals, we identified clusters of particularly consequential meanings and reasoning, based on comments in the literature and our own judgment as teachers and researchers. We selected those involving accumulation and area for our first round of developing assessment items. We collected variations of conceptual questions from the literature and available homework and exams then selected and refined seven of these questions that we felt best addressed the meanings and reasoning identified in the literature. We then developed interview protocols which included contingencies for anticipated student responses.

We conducted and video-recorded one-on-one interviews with 12 students at three institutions. These students were either completing a first-semester calculus course or currently enrolled in a second-semester calculus course. The interviews were approximately 90 minutes in duration and consisted of the seven open-ended questions with ample space for students to show their work. Each interviewer followed the protocols and encouraged students to elaborate their meanings and reasoning. In cases with unexpected responses, the interviewer probed for additional reasoning or allowed the student's reasoning to take its course then provided additional information or suggestions to see how they would proceed. We created detailed summaries of the students' work with notes about potential underlying meanings and reasoning. The entire research team then met to discuss the notes and create a model of productive understandings and the meanings with substantial evidence in the data.

Results

In this paper we discuss and contrast two pairs of tasks. The first pair (Tasks 1 and 2) requires an MBS and a QBS conception of the definite integral, respectively. The second pair (Tasks 3 and 5) illustrates students' conceptions and reasonings about area when interpreting integrals given a graphical representation.

Task 1

Carl was driving on the highway when he saw a sign that said ROAD ENDS 250 FEET. He slammed on his brakes and came to a stop in 4 seconds. Below is a table of data from his truck's computer while he was braking.

Time (seconds)	0	0.5	1	1.5	2	2.5	3	3.5	4
Speed (ft/s)	98	96	92	84	74	60	43	23	0

Given the table, what is your best estimate for the total distance Carl traveled before coming to a stop?

Most students approached this problem heuristically, drawing the graph of a function and finding the area under it. They often reached a correct conclusion using memorized procedures and associations rather than expressing any meaningful quantitative reasoning. Some applied a simple unit analysis to verify their computations but did not support this analysis with associated quantitative reasoning. Others, with prompting, did eventually invoke some quantitative meanings, which however did not alter their approach or answer.

Student 1: Multiply…feet per second…times one second. How much the area is underneath that [graph]…that's how many…feet.

$$0.5(\cancel{98}+96+92+84+74+60+43+23+\cancel{10})$$

Student 2: The speed, feet per second times second is just feet. All of this times .5.

Student 3: Speed is feet a second and time is just measured in seconds, so if I multiply time times speed, it can obviously give me a distance. Then obviously, the value under the curve is the entire…estimate.

Task 2

A train travels along the monorail at a theme park and is programmed to change speed according to where it is on the track, to make turns, stops, etc. The train passes a sign that says the next stop will be in 30 meters. The speed of the train is $v(x)$ meters per second at a point x meters along the track from the sign. As it approaches one stop, its speeds at 5-meter increments are given in the table below.

Distance from sign (m)	0	5	10	15	20	25	30
Speed (m/s)	8	6	5	4	3	2	0

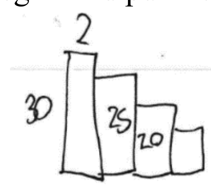
Given the table, what is your best estimate for how long it takes the train to stop from 30 meters out?

Some students guessed the correct integral using unit analysis. Without the underlying quantitative reasoning that is necessary in a problem such as this, however, they were unable to make accurate associations and generate the appropriate Riemann sum. This discrepancy is partly due to their perception that the answer should be the area under whatever curve they have.

Student 1: [manipulates formula and tries switching axes and using slope] Slope is rise over run. You could take the integral and put it under 1. You can just set distance as the *y*..

$T = \frac{D}{S}$

$\text{Slope} = 1/T$



$$2(30+25+20+15+10+5)$$

Student 2: [using non-quantitative unit analysis and manipulating ratios]: I'm multiplying distance times speed and that's meters times m/s…but distance times speed does not equal…time. If 8 is to 1, then what is that to 5 over *x*.

$m \cdot m/s$

$\frac{m}{m/s}$ $\quad m \cdot \frac{s}{m}$

$\frac{8}{1} = \frac{5}{x}$ $\quad 8x = 5$ $\quad 0.625\,s$

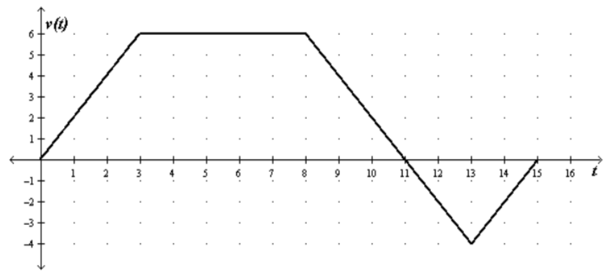
$0.625 + 5/6 + 1 + 5/4 + 5/3 + 5/2 = 7.875\,s$

Task 1 vs. Task 2

Most students approached Task 1 procedurally without any quantitative reasoning about (a) the distance-velocity-time relationship or (b) the structure of a definite integral. This type of problem invoked an instant association with computing a Δt, multiplying by function values, adding the results, areas of rectangles on a graph, area under the graph, and an integral representing area and/or distance. However, there was no connection between these associations and meaningful quantitative reasoning. In contrast, Task 2 required students to reason quantitatively in the context of the problem rather than simply use formula manipulation and/or unit analysis. Those who successfully completed the task, developed a QBS interpretation that enabled them to break up the scenario into pieces of time and create an appropriate Riemann sum. Further, these correct answers differed from their initial procedural approach.

Task 3

Assume that the graph of $v(t)$ shown below represents the velocity of a car (in meters per second) that is traveling along a straight road. Positive velocity corresponds to traveling straight north.



1. How far is the car from its starting point after 15 seconds? Explain.
2. How far does the car travel over the 15-second interval? Explain.
3. Write down integrals that correspond to the numerical answers for #1 and #2.
4. Compute $\int_0^{15} v(t)dt$.
5. Is there an interval on which $\int_0^{15} v(t)dt \int_a^b v(t)dt = 0$? If so, where, and what does it mean in terms of the context of the car?

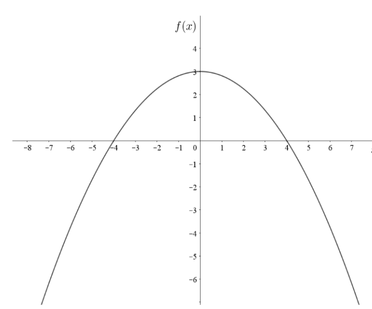
Draw a graph of position vs. time and relate it to each interval of the velocity vs. time graph.

Most students were able to interpret the movement of the car correctly. However, when finding displacement or distance traveled, many relied on memorized procedures with no productive understanding of the situation.

Student 1: I'm not even really reading the words. I'm just seeing graphs. Find area.
Interviewer: Does it matter that velocity is the *y*-axis and time is the *x*-axis?
Student 1: No. That helped, I suppose.
Student 2: Why would I set up an integral when I could just like use shapes?
Student 3: I just found the difference of those areas.
Interviewer: What led you to compute areas?
Student 3: That was just my go-to…start finding the areas. I just remembered doing it from other problems.

Task 5

Use the graph below for this item. For each pair below, determine which one is larger or if the quantities are equal. Explain how you know.



a) $\int_0^2 f(x)dx$ _____ $\int_0^4 f(x)dx$ (fill in the blank with <, >, or =)
b) $\int_3^4 f(x)dx$ _____ $\int_4^6 f(x)dx$ (fill in the blank with <, >, or =)
c) $\int_0^4 f(x)dx$ _____ A left-hand Riemann sum on the interval from *x*=0 to *x*=4. (fill in the blank with <, >, or =)

Determine if each expression below is positive, negative, or zero. You do not need to evaluate the expression. Explain how you know.

d) $\int_{-4}^{4} f(x)dx$ e) $\int_{-3}^{3} f'(x)dx$ f) $\int_{3}^{1} f(x)dx$

This question revealed flawed reasoning that can result when students lack a meaningful basis for their understandings. Imagistic reasoning led some students to misinterpretations.

Student 1: [explaining his correct answer (>) on part (b)] 4 to 6 isn't even above the x-axis, I don't think it makes it negative. It's still area. I just put that because since 3 to 4 was above the x-axis, it'd be positive, and this 4 to 6 wouldn't really count in my mind…so I just discredited it. If you do think about if it keeps going below the x-axis, it would be…infinite. If you're just thinking about it being below this graph, it keeps going down.

Student 2: [explaining his incorrect answer of "0" on part (d)] Assuming it's a symmetrical parabola, the negative area [points to left of y-axis] is equal to the positive area [points to right of y-axis], so they cancel each other out.

Task 3 vs. Task 5

The use of imagistic and associative reasoning enabled some students to answer question 3 correctly but caused serious interpretive problems in question 5. For example, student 1 (above) conceptualized the area below the curve extending to negative infinity yet obtained the correct answer to question 3 (with bounded regions). Student 1 seemed to think that all area is positive and "discredited" the area that was below the x-axis.

Discussion

Although we may be tempted to be complacent about students' memorized associations and procedures, these questions reveal that when not backed by productive understandings of the underlying mathematical concepts, they are merely arbitrary rules. Without a meaningful basis, students are likely to invoke highly inappropriate variations of these procedures alongside or even displacing appropriate ones.

We reviewed Tallman et al.'s (2021) analysis of productive meanings assessed on calculus exams in the United States. Results showed that the exams generally required low levels of cognitive demand and students were rarely required to demonstrate or apply their understanding of concepts. Therefore, the need arises for exams to contain discerning questions to assess students' quantitative conceptions. To facilitate this assessment, students must be required to model the accumulation of quantities using a localized adaptation of a basic model relating the multiple relevant quantities. They must also explicitly connect definite integrals with the approximations generated from the resulting Riemann sums.

Ultimately, we aim to develop an instrument that will (a) identify aspects of productive understandings that are both broadly and effectively adopted by students to make sense of calculus content, (b) evaluate broad-scale interventions which seek to identify instructional practices that lead to productive understandings of calculus content, and (c) convey important characteristics of productive understandings to the general mathematics faculty in order to illuminate effective and ineffective instructional practices in calculus.

References

American Educational Research Association. (2014). *Standards for educational and psychological testing.* Washington, DC: American Educational Research Association.

Gleason, J., Thomas, M., Bagley, S., Rice, L., White, D., and Clements, N. (2015) Analyzing the Calculus Concept Inventory: Content Validity, Internal Structure Validity, and Reliability

Analysis, In *Proceedings of the 37th International Conference of the North American Chapter of the Psychology of Mathematics Education,* 1291-1297.
Epstein, J. (2013). The calculus concept inventory – Measurement of the effect of teaching methodology in mathematics. *Notices of the American Mathematical Society, 60*(8), 2-10.
Jones, S. R. (2015). Areas, anti-derivatives, and adding-up-pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior, 38*, 9-28.
Meredith, D. C., & Marrongelle, K. A. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics, 76*(6), 570-578.
Oehrtman, M. & Chhetri, K. (2015). The equation has particles! How calculus students construct definite integral models. In *Proceedings of the 18th annual conference on research in undergraduate mathematics education* (pp. 19 - 25). Pittsburgh, PA.
Reed, Z., Tallman, M.A., Oehrtman, M., & Carlson, M.P. (2022) Characteristics of Conceptual Assessment Items in Calculus. *PRIMUS*, *32*(8), 881-901. DOI:10.1080/10511970.2021.1954114.
Reed, Z., Tallman, M.A., Oehrtman, M., & Carlson, M.P. (under review). Writing exam items that assess productive meanings in calculus.
Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient. In S. Alatorre, J. L. Cortina, M. Sáiz, y A. Méndez (Eds.), In *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 46). Mérida, México: Universidad Pedagógica Nacional.
Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*, 230-245.
Simmons, C. & Oehrtman, M. (2017) Beyond the Product Structure for Definite Integrals. In *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 912-919). San Diego, CA:
Tallman, M.A., Reed, Z., Oehrtman, M., & Carlson, M.P. (2021). What meanings are assessed in collegiate calculus in the United States? *ZDM-Mathematics Education, 53*, 577-589.
Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics-Physics Education Research, 8*(1), 010125.