Beliefs and Rationales For Instructor Questioning Choices During Class Discussion of the First Isomorphism Theorem

Lino Guajardo
Texas State University

Kathleen Melhuish
Texas State University

Holly Zolt
Texas State University

Mario Gonzalez
Texas State University

Anthony Tucci
Texas State University

Cody Patterson
Texas State University

Eugene Curtin
Texas State University

Previous studies have established teacher questions as a significant interaction component in proof-based courses whether lecture-based or student-centered. However, little research has been conducted to investigate behind the types of questions asked in class. In this report, we share an analysis of three instructors' lessons on the First Isomorphism Theorem. We characterized instructor question types and then conducted interviews with each instructor to investigate how their pedagogical beliefs relate to the types of questions they ask in class. We found that the instructors held similar pedagogical beliefs about student learning, but diverged in their views on the purpose of the course. Both beliefs on learning and their beliefs on purposes of the course linked to the nature of the questions asked in class.

Keywords: Instructor beliefs, instructor questioning, proof presentation

Instructor questioning is one of the primary ways students are invited to participate in mathematical proof courses (Artemeva & Fox, 2011, Melhuish et al., 2022; Paoletti et al., 2018). We conjecture that the types of questions instructors asked are shaped by their beliefs about teaching, learning, and proof-based courses. While we know broad information about instructors' pedagogical decisions, investigations into the how and why of instructional practices in proof-based courses are still needed.

In this paper, we set out to address the research question: how do abstract algebra instructors' pedagogical beliefs relate to the types of questions they ask in class? In proof-based courses, research on teaching practices can be sorted into two categories: lecture style teaching and student-centered teaching (Melhuish et al. 2020). Analysis of teaching actions, such as types of questions asked, is largely limited to lecture settings where instructors frequently engage in "chalk talk" (Artemeva & Fox, 2011) writing proofs on the board, adding verbal commentary, and asking questions. In contrast, in student-centered instruction studies, most focus has been on student activity and tasks with the role of the instructor backgrounded. There has also been some work has focused on the mathematical and noticing activity of instructors and how these shape in-the-moment instruction (e.g., Johnson and Larsen, 2012; Johnson 2013). With these different foci, several questions are open including the nature of questioning in more active classes, and rationales for how and when instructors ask questions across both settings. This study makes a comparison between instructors who teach a lecture style classroom versus those who teach a student-centered classroom. Specifically, we plan to investigate the way that instructor pedagogical beliefs shape the questions asked in class. For most classes, questions are one of the most common ways for students to be invited to participate in the class (c.f., Melhuish et al., 2022; Paoletti et al., 2018).

Relevant Literature

Researchers have investigated instructor actions in lecture and rationales for their teaching styles. Paoletti et al. (2018) found that instructors typically ask factual and next step type questions (see Table 1 for definitions question types) during lectures in proof-based classes. Fukawa-Connelly et al. (2016) surveyed mathematicians and found that lecturers chose the style because of concerns about time and covering material. Woods and Weber (2020) reported similar findings. They also discussed other goals and orientations of lecturing instructors and how these can affect the instructor's in-class behaviors such as lecture allowing for mathematical precision and formalism. That is, the literature largely suggests that lecturers are intentional and have robust rationales for the way they teach (Melhuish et al., 2022).

Instructors also diverge on views of advanced courses and what the overall emphasis should be (e.g., Alcock, 2010). For example, Johnson et al. (2013) conducted interviews with mathematicians implementing an inquiry-oriented (IO) abstract algebra curriculum and found that some instructors held different goals for the abstract algebra course (with or without the IO curriculum they were implementing). She found that one of her instructors emphasized mathematical practice and activities, while the other instructors had a heavier focus on the abstract algebra content. Each of these instructors had beliefs about what students should gain and learn from the course and then implemented and focused the course on these beliefs, even when using the IO curriculum.

A recent case study was conducted by Rupnow (2021) where instructors were interviewed on their beliefs about mathematics instruction and their abstract algebra classes were observed to determine how these beliefs were enacted. In terms of questioning, Rupnow observed that the IO instructor asked more questions than the lecturer. Rupnow reported on two instructors, one using an IO curriculum where students were more actively working on mathematics in class while the other instructor more traditionally lectured. The instructors' beliefs impacted the structures of the class and these structures changed over time due to additional beliefs such as those related to coverage.

Conceptual Framing

The framing underlying our study draws on Leatham's (2006) sensible system framework. In this framework, teachers are viewed as sensible actors who can and do hold many beliefs. These beliefs work together in a sensible system, with some holding more power than others and some beliefs leading to other beliefs. A teacher's belief may seem contradictory to an observer's point of view, but from the teacher's point of view all their beliefs are consistent and work together. Some relevant beliefs may reflect orientations towards teaching, learning, and course goals.

We then make the assumption that beliefs play a mediating role in the nature of questions asked in classrooms. To parse question types, we adopt Paoletti et al.'s (2018) analytic framework (Table 1) of types of questions that were asked by instructors in proof-based classrooms.

Table 1: Paoletti et al.'s (2018) Question Type Framework

Question Type	Definition
Fact	Questions that elicit a closed form mathematical response. These questions do not call for a course of action from the students.
Next Step	Questions that elicit recommendations from students on what to do next within a proof or example. The purpose of these questions is to create logical progressions.
Proof Framework	Questions that elicit higher-order levels of proof structure. Such questions targeted proof structure such as type of proof or how to generically prove a construct such as one-to-one.
Warrant	Questions that elicit justification responses to a statement or claim.
Evaluation	Questions that elicit the truth-value of a statement. Typically answered with a yes or a no.
Convention	Questions that elicit mathematical convention of proof writing or notation.
Other	Questions that do not fit into any of the other question types.

Methods

Data for this study was collected for a larger NSF-funded design-based research project[1]. Data were collected in three instructors' abstract algebra classes at a large research institution in the United States on days they covered three key theorems in the class. This paper focuses on the lessons addressing the First Isomorphism Theorem (FIT). Instructors A and B used an inquiry-oriented approach to help students in comprehending the FIT. Instructor A was involved in the creation of the materials, and both Instructors A and B used these materials in their classes. This lesson was implemented across two 80-minute (160 minutes total) class days, with each instructor taking a total of approximately 61 and 65 minutes, respectively, of whole class instruction. Instructor C covered the FIT via an interactive lecture in one 80-minute class day with their lesson taking 54 minutes. All instructors have had prior experience teaching abstract algebra and had strong student evaluations. Instructor A has a Mathematics Education Ph.D., and their primary area of research is teaching and learning in advanced proof classes. Instructor B holds a Mathematics Ph.D. with a focus in geometric group theory but has transitioned to mathematics education research. Instructor C holds a Mathematics Ph.D., and his primary area of research is graph theory and combinatorics. Focal lessons from each instructor were video recorded and transcribed. At the end of the academic year, semi-structured stimulated recall interviews (Schepens et al., 2007) were conducted with each instructor. For each participant, a member of the research team went through the transcript data of classroom observations and identified all questions that met the following two criteria: (a) the question had to have been posed to the whole class and (b) the question had to have adequate wait time of at least 3 seconds. The first criterion was implemented so we could analyze the ways in which instructors ask questions posed to the class and how these questions may reflect their beliefs. The second criterion was established to rule out questions where the instructor posed the question and immediately answered it. Questions that were asked towards individual students or groups of students during small group discussions or independent thinking were not considered for this analysis.

[1] DUE# 1836559

Once all questions asked by the instructor had been identified, the research team independently went through each of the participants' questions to identify a representative set of questions. That is questions of a similar type (such as the multiple instances where instructors would ask students for the definition of a concept). The research team then met to determine which questions would be part of the stimulated recall interviews. Instructor A was interviewed first by the whole research team and the data was initially used as a pilot for interview protocol question creation and interview practice. Two members of the research team interviewed Instructor B and two other members of the research team interviewed Instructor C.

In addition to having instructors provide rationales for their questions, they were also prompted to answer questions about how students learn, the purpose of the course, and the meaning of mathematics. The interviews were then analyzed by two research team members independently to open code for salient instructor beliefs. The questions asked during the lesson were coded using the above analytic framework by two members of the research team. Any discrepancies in coding were resolved through discussion. After completion of this initial analysis, the interview transcripts and question rationales were considered in light of the types of questions asked, instructor beliefs, and the rationales provided by the instructor to develop initial explanatory mechanisms for the types of questions asked.

Results

To contextualize our results, we provide the frequency of types of questions in Table 2. Note that Instructor C's question profile is reminiscent of most lecturers from Paoletti et al.'s (2018) study. Instructor C asked mainly fact questions (63%[2]) with next step as the second most frequent (19%). Instructor A asked "other" questions most frequently (43%) and fact questions (40%) as the second most frequently asked question type. While Instructor B asked fact and other types questions equally frequently with both having the highest frequency in comparison to the remaining question types (34%). We also note that unlike in Rupnow (2021), the number of questions asked in the lecture class and the active classes were not substantially different. Rather, we see that the type of questions asked in the inquiry proof comprehension lessons were qualitatively different than those asked in the lecture class on the same theorem.

Table 2: Frequency of question types

Instructor	Facts	Next Step	Proof Framework	Warrant	Evaluation	Convention	Other	Total
Instructor A	23	0	3	5	2	0	25	58
Instructor B	23	3	4	5	5	4	23	67
Instructor C	39	12	6	1	2	2	0	62
Total	87	15	13	11	9	6	45	186

[2] All displayed percentages are rounded percentages of total questions asked by that instructor

In order to make sense of these qualitative differences, we turn to the instructors' stated beliefs. All three instructors shared similar ideas about how students learn focusing on students being actively engaged in the learning process. Instructor A explained that students need to be "actively engaging in things by making connections to prior knowledge" elaborating that this is hands-on and not just mimicry. Instructor A elaborated that students "need to have opportunities to really think through ideas." Instructor B explained that experience and problem solving are the most important things for students to learn, stating "you learn through problem solving." Adding that to accomplish this, Instructor B tries "to set-up a learning experience so there are opportunities to be active where being active will lead to at least some degree of success; some degree of intuition building." Instructor C stated that, "You learn by doing." For Instructor C, they viewed this idea of doing by having students engage in proof production, whether in-class by answering questions or by reproducing proofs on their own that they may have already seen. Each of these instructors viewed similar notions about learning in having students be active in their courses and attended to providing opportunities for students to engage in the course.

While the instructors provided similar ideas about learning, they diverged on their primary foci in describing the aims of the course. All instructors had a shared focus on advanced mathematical content and thinking more broadly with Instructor A discussing apprenticeship into research mathematics, Instructor B stating that 90% of the course is "what it does for you as a mathematics learner" and Instructor C explaining the course as an "introduction to more advanced mathematics." The instructors also all valued abstract algebra content-specific aims, such as better understanding underlying mathematical systems and structures students have seen. However, the respective weights and interpretations of each component differed. Throughout the interview, Instructor A tended to focus on understanding the concepts specific to abstract algebra, Instructor B focused on the problem-solving involved in proving and sense-making, and Instructor C focused on developing knowledge to succeed in constructing formal proofs.

We believe the similarities and differences in beliefs, as expressed in the responses to how students learn and the aims of the course play a large role in the questions asked in class. All three instructors believe students should be active in their learning, and this manifests in different ways through the questions. Further, we hypothesize that the belief in the purpose of the course also interacts with the questions asked by further pushing their narratives in what the students should be focused on when discussing or contributing to the content.

Throughout the interview with Instructor A, we saw many references to the importance of having students wrestle with complex ideas within class and a focus on abstract algebra specific content. For example, when asked about the purpose of turn-and-talks (a frequent occurrence in Instructor A's class), Instructor A stated that they want to get "more students actively involved in the conversation. And give them a chance to tackle something that is kind of complicated." For example, Instructor A gave students question prompts such as, "What in the world is a quotient group?... And I don't care if it's a formal definition or an informal explanation, but what is a quotient group for us?" (Other type question). This question had students actively engaging in their learning but also focuses them on the abstract algebra content of the course. Instructor A provided further rationale that they choose to ask questions about content-related issues that might be "stumbling blocks" for students. For example, Instructor A prompted another turn-and-talk sequence with "What is the image of G? What do we mean by that? What is that thing representing?" (Other type question). This prompt focused students on attending to the difference between phi as a function and phi(G) being the image of a function. They chose the turn-and-talk mechanism to specifically give more students opportunity to talk and actively develop their conceptual understanding. This reflects the stated value of having all students draw on their knowledge and participate in making sense of complex ideas.

Unlike Instructor A, Instructor B was using curriculum materials that were developed externally. His value on problem-solving led him to note that without this task he may have "handed off actually more of it to the students" in terms of constructing the proof rather than just comprehending it. Instructor B's focus on process and problem-solving came through via questions even though his lesson contained many of the same features as Instructor A's. For example, Instructor B prompted his students to construct an isomorphism map that takes a coset to its image (first requesting what to call the input (convention question) then where it might map to (next step question)). He noted, "I kind of wanted students to experience coming up with that." Instructor B further added that this also gave students more agency, stating "I feel like if I can hand a little bit of that agency of coming up with the map off the students, then then they own a little bit more of that now." A student then suggested mapping to the representative element ($\beta(aK) = a$). Instructor B then asked, "So what might be problematic about saying this image is going to be just lowercase a here?", an *other* type of question. Instructor B is encouraging students to engage in analyzing their mistakes and rectifying, a key part of problem-solving. Another example is during the student presentations of each portion of the FIT proof. Instructor B gave students the prompts "focus on helping the class understand why this stuff is happening. Why did they do this?" Instructor B focused on problem-solving and processes, allowing students to be the owners of their learning and frequently putting them into situations in which they can productively struggle with the concepts and the proving process.

For Instructor C, many of his questions asked for next steps and facts during the proving process. These questions were asked to engage students in the production of the proof. He explained that "students will tune out if they don't feel ownership in what you're doing right, so that's certainly… if I am going to do something, I want them to be interacting with me." Instructor C further elaborated that in an instance of a complex theorem, such as the FIT, his role is to break the proof into "bite sized pieces" for students to develop understanding of the idea in each of the pieces. For example, Instructor C asked the proof framework question "What else do we have to do?" This question had students attend to the different pieces of the FIT proof, specifically the different pieces of showing a mapping is an isomorphism. Instructor C also modeled the types of actions students would want to take in their own proving endeavors by asking next steps questions ("How can I use that ψ is onto?") and for definitions ("So what does ψ from A to B being a homomorphism mean?") to contribute to the proof construction. Instructor C noted that one of the major goals of the course is for students to use definitions to structure proofs and that students struggle with this idea. Instructor C's way of creating active learning is to engage students by asking small, manageable questions for the students to handle while he supports them in using the formal proof system, definitions, and attending to appropriate precision.

Discussion

We selected a common lesson focus across three instructors to better understand the nature of questions asked in an advanced undergraduate class and ultimately the rationales for selection of questions. Two of the instructors implemented an inquiry-oriented lesson where students engaged in comprehending the proof of the First Isomorphism Theorem. The third instructor presented the theorem and constructed its proof through an interactive lecture. Our first contribution is illustrating that the three instructors had many commonalities in terms of how they believed students learn. All instructors discussed students being actively involved and doing mathematics. Furthermore, throughout the interviews, all three instructors talked about students taking ownership over the mathematics in the class. However, we can note that the type of activity asked of the student varied. Instructor B had much of the activity divided between group work and whole class instruction with a focus on what students could construct or problem-solve. Instructor A used a combination of group work, but also a turn-and-talk strategy where students engaged with partners briefly before responding to whole class discussion questions about points of difficulty related to the abstract algebra content. Instructor C used interactive discussion focusing on targeted questions that allowed students to contribute to the construction of the proof while also modeling ways to structure a proof based on using formal definition.

As noted in Melhuish et al.'s (2022) literature synthesis, the choices that instructors make in advanced classes are well grounded and rational in view of pedagogical goals. Instructor C provides additional insight into the results of Paoletti et al.'s (2018) paper on lecture instruction in proof-based courses. The questions Instructor C asked were in service of students being both actively engaged in the proof construction process, while also being constrained enough such that the instructor can model nuances in this new way of thinking: using definitions to construct proofs.

In contrast, Instructor A and B would have been considered anomalies in terms of their question profiles; however, quite similar to each other. However, where we saw substantial differences was in their questions in the "other" bin. Instructor A frequently used their knowledge of where students might struggle with abstract algebra ideas and notation to provide students time to recall prior knowledge and use their own words to make sense of ideas. Instructor B commonly focused on elements of problem-solving and letting students construct components where possible. These results contribute to the larger narrative about instruction in inquiry settings and surface somewhat distinct questioning profiles.

The overarching beliefs about course purpose provide some insight into these differences with Instructor A focusing heavily on abstract algebra content, Instructor B on the problem-solving process, and Instructor C on formal proofs and their connection to definitions. If we turn to Alcock's (2010) identification of four modes of thinking in proof classes, we could suggest that Instructor C tended towards syntactic thinking (generating proofs of statements by formal structure), Instructor A towards semantic thinking (developing understanding of a statement via attention to concepts), and Instructor B towards creative and critical thinking (examining objects to support proving and checking correctness.) Of course, any such classification would oversimplify the instruction and instructor rationales as elements of each thinking can be identified across instructors. Overall, we suggest that these orientations may play a more substantial role in questioning types in an advanced class than more general beliefs about learning and mathematics which tended to be rather uniform (at least from the surface) for our three instructors.

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