Students' Justifications and Their Proofs by Mathematical Induction

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Mathematical induction has proven to be difficult for students to understand (Ernest, 1984) and, more broadly, the purpose of proof; often remaining unfulfilled as to the certainty of $P(n)$ for values of n for which the statement is proven (Avital & Libeskind, 1978; Harel, 2002). The research perspective in this paper is the proving-as-convincing research paradigm (Stylianides et al., 2017).

Some studies have focused on students' convictions with how they develop or shift (Brown, 2014) and how students convictions may be understood (Weber & Mejia-Ramos, 2015). Others have studied how mathematical arguments may be convincing (Stylianides & Stylianides, 2009). Generally, students may show evidence of this with their justifications regarding proofs.

As part of a larger study, we had students complete a task asking to produce a proof by mathematical induction and provide justifications for the truth value of $P(n)$ for certain values of n. The purpose of this study is to examine the relationship between students' abilities to construct a proof by mathematical induction and their types of justifications based on their written responses to the task. In this study, 59 undergraduate STEM majors across two semesters were given the following task:

a) Prove the statement $P(n)$ by mathematical induction for $n \in \mathbb{N}$, $n \geq 6$,
$P(n)$: $17 + 20 + 23 + \ldots + (3n - 1) = n(3n + 1)/2 - 40$.
b) Is the statement true for $n = 10$? Justify your answer.
c) Is the statement true for $n = 5$? Justify your answer.
d) Comment on your difficulties, if any.

For a preliminary analysis, a framework has been developed loosely based on work by Harel and Sowder (1998) who described the act of justifying as "how students ascertain for themselves or persuade others of the truth of a mathematical observation" (p. 243). We have incorporated the domain of discourse (Stylianides et al., 2007) and domain of validity (Healy & Hoyles, 2000) into a framework to help define these terms.

For proof justification, a student provides justification using his or her proof of the statement. For $P(10)$, it is inferred that the student understands the associated domain of validity for his or her proof. For $P(5)$, it is inferred that the student is convinced that for any value not within the domain of validity (but in the domain of discourse), the statement is false. For empirical justification, the student does not use his or her proof for justification. However, a student may feel the need to provide an additional verification, such as a calculation, that is different from his or her proof after having already proven the statement.

We found that students often relied on empirical justifications (53%), as reported in the literature. However, we also found interesting results. Some students (17%) relied on the proof for both cases, and a few (7%) used proof justification for $P(5)$ and empirical justification for $P(10)$, a result found only to be reported once in the literature (Stylianides et al., 2007). These findings contribute to the field by showing that students may use different sources for justifying statements depending on where the value in question lies. Educators may use these results to attend to how students may operate in the domain of validity and domain of discourse when understanding the purpose of proof.

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Students' Justifications and Their Proofs by Mathematical Induction

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Rationale

Mathematical induction has proven to be difficult for students to understand (Ernest, 1984) and, more broadly, the purpose of proof; often remaining unfulfilled as to the certainty of *P*(*n*) for values of *n* for which the statement is proven (Avital & Libeskind, 1978; Harel, 2002).

The research perspective in this paper is the proving as convincing research paradigm (Stylianides et al., 2017). Stylianides et al. (2017) stated that this research paradigm on proof revolves around studying students' convictions with evidence (i.e., mathematical arguments and proofs) and how it is related to the mathematics community's accepted forms of proof.

In this proving as convincing research paradigm, deductive arguments, such as proofs by mathematical induction, convince mathematicians while arguments relying on specific calculations or having the appearance of proof remain unconvincing (Stylianides et al., 2017).

Theoretical Background

Ernest (1984) described the necessary steps to prove a statement by mathematical induction which grounds itself in *modus ponens*, a deductive argument, and the Principle of Mathematical Induction (PMI).

For mathematical statements, the domain for finding whether a statement is true is called the domain of discourse (Stylianides et al., 2007). The domain of discourse for proofs by mathematical induction is any natural number, *n*, greater than the basis step, n_0 ($n \geq n_0$). In Healy and Hoyles (2000), they called the domain for which a statement is known to be true the domain of validity. The domain of discourse for *P*(*n*) in the task for this study is $n \geq 5$ where *n* is a natural number, however, the domain of validity is $n \geq 6$.

For this study, the term justification describes how a student determines the validity of a mathematical statement for a value (Harel & Sowder, 1998). Students provide these justifications with written responses to the task. Note that the proofs with which students interact are their own proofs by mathematical induction.

We have incorporated the domain of discourse (Stylianides et al., 2007) and domain of validity (Healy & Hoyles, 2000) into a framework to help define these terms.

Categories

For proof justification, a student provides justification using his or her proof of the statement. For *P*(10), it is inferred that the student understands the associated domain of validity for his or her proof. For *P*(5), it is inferred that the student is convinced that for any value not within the domain of validity (but in the domain of discourse), the statement is false.

For empirical justification, the student does not use his or her proof for justification. However, a student may feel the need to provide an additional verification, such as a calculation, that is different from his or her proof after having already proven the statement.

Purpose

As part of a larger study, we had students complete a task asking to produce a proof by mathematical induction and provide justifications for the truth value of *P*(*n*) for certain values of *n*. For these values, we investigated students' responses for which source, either their proofs or non-proof related reason(s), was used to determine the truth of the statement.

The purpose of this study is to examine the relationship between students' abilities to construct a proof by mathematical induction and their types of justifications based on their responses to the task.

The Task

In this study, 59 undergraduate STEM majors were given the task below. With the students' responses, we performed preliminary analyses of the justifications, and we discuss these students' justifications.

a) Prove the statement *P*(*n*) by mathematical induction for $n \in N, n \geq 6$,

$$P(n)\text{: } 17 + 20 + 23 + \ldots + (3n - 1) = \frac{n(3n+1)}{2} - 40.$$

b) Is the statement true for $n = 10$? Justify your answer.

c) Is the statement true for $n = 5$? Justify your answer.

d) Comment on your difficulties, if any.

Table 1. The interactions between the responses of whether P(10) and P(5) are true.

	Evidence of Source	*P*(10) result: Proof Justification	*P*(10) result: Empirical Justification
P(5) result	Proof Justification	17 (29%)	4 (7%)
	Empirical Justification	7 (12%)	31 (53%)

(Top Left) Proof Justification for P(10) and Proof Justification for P(5).

b) Is the statement true for n=10? Justify your answer.

Yes, the statement is true for n values greater than or equal to 6. Since 10 is greater than 6, the statement holds true.

c) Is the statement true for n=5? Justify your answer.

No, the statement is not true for n = 5. The statement is only true when n = 6 or greater than 6. Since 5 is a lower number than 6, the statement is not true for when n = 5.

(Top Right) Empirical Justification for P(10) and Proof Justification for P(5).

b) Is the statement true for n=10? Justify your answer.

$\frac{10(3(10)+1)}{2} - 40$ $\quad 3(10)-1=29 \quad 155 - 40 = 115 \quad 29 \neq 115$

b) is false

n=10 is false

c) Is the statement true for n=5? Justify your answer.

no because $n \geq 6$

$5 \leq 6$ which makes this false

(Bottom Left) Proof Justification for P(10) and Empirical Justification for P(5).

b) Is the statement true for n=10? Justify your answer.

Yes because we checked and the statement is true for any $n \geq 6$ by the PMI, and $10 \geq 6$

c) Is the statement true for n=5? Justify your answer.

It is not true for n=5.

$P(5)\text{: } 3(5)-1 = \frac{5(3(5)+1)}{2} - 40$

$14 \neq 0$

(Bottom Right) Empirical Justification for both P(5) and P(10).

b) Is the statement true for n=10? Justify your answer.

$\frac{10(31)}{2} - 40 =$

$155 - 40 = 115$

$17+20+23+26+29$

$37 + 49 + 29$

$86 + 29 = 115$

$n = 10$ ✓

I was thinking too fast and I forgot the −40

c) Is the statement true for n=5? Justify your answer.

$(3(5)-1) = 14$

$\frac{5(16)}{2} - 40$

$\frac{80}{2} = 40 - 40 = 0$

$14 \neq 0 \quad n=5$ not true

Findings

Note that students' written comments on part d) of the task are included in the figures. Some students declined to respond to part d).

We found that students often relied on empirical justifications (53%), as reported in the literature. However, we also found interesting results. Some students (17%) relied on the proof for both cases, and a few (7%) used proof justification for *P*(5) and empirical justification for *P*(10), a result found only to be reported once in the literature (Stylianides et al., 2007).

Some students responded to part d) by writing that they did not experience difficulties. This is interpreted that students may not perceive the relationship between a proof and the domains of discourse and validity.

These findings contribute to the field by showing that students may use different sources for justifying statements depending on where the value in question lies. Educators may use these results to attend to how students may operate in the domains of discourse and validity when understanding the purpose(s) of proof.

Next Steps

We plan to use these findings to help develop tasks to better understand how students' justifications change based on the domains of discourse and validity.

Future research investigations may include these constructs and the relationship with students' convictions with mathematical statements and arguments.

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