Calculus I Instructors' Use of Representations in Instructional Tasks: Introducing Derivatives with Inquiry

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I present preliminary results from my dissertation research that investigates how eight college Calculus I instructors used representations in an instructional task intended for introducing derivatives with inquiry. Using Zandieh's (2000) framework, the instructors were asked to propose and sequence eight tasks for introducing derivatives to their students. This paper focuses on the instructors' utilization of various representations of derivatives in the first task they proposed in the sequence. The findings show how all instructors found graphical and physical representations appropriate for introducing derivatives, with some of them also reaching out to other representations (symbolic, numerical, and verbal) simultaneously.

Keywords: Instructional tasks, derivatives, representations, inquiry, calculus teaching

Research on the impacts of inquiry teaching suggests positive student outcomes, such as learning and persistence in STEM (Freeman et al., 2014; Kogan & Laursen, 2014; Laursen et al., 2014; Johnson et al., 2019; Rasmussen & Ellis, 2013). It is also known that the implementation of inquiry methods varies considerably among instructors (Laursen & Rasmussen, 2019). Research on the variability of inquiry-oriented teaching in undergraduate mathematics has mostly focused on characterizing and measuring how instructors and students interact in the classroom (e.g., Laursen et al., 2011; Mesa et al., 2020; Shultz, 2020). While showing the range of pedagogical variability in these classrooms is necessary for understanding inquiry teaching, this strand of research overlooks the content instructors and students interact with.

In my dissertation study, I investigate how college calculus I instructors, who use inquiry in their teaching, shape students' mathematical work during instruction of one specific piece of content: derivatives. As one of the basic and core concepts in calculus, derivatives' multiple representations and conceptualizations make its teaching a rich area for investigation. I bring my attention to the content during instruction by looking into the instructional tasks and their sequencing that instructors use to introduce derivatives to their students. For this preliminary report, I focus on instructors' use of mathematical representations, specifically addressing the following research questions:

1. Which representations do college Calculus I instructors, who teach with various inquiry approaches, use within instructional tasks to introduce derivatives with?
2. What patterns exist, if any, in their use of representations in instructional tasks for introducing derivatives?

Research on Representations of the Concept of Derivative

Research on representations of the concept of derivative has mostly attended to student understanding via using various representations, making connections among representations, and extending student understanding to applications of derivatives in other contexts (e.g., Feudel & Biehler, 2021; Jones & Watson, 2018; Maharaj, 2013; Roorda et al., 2007; Siyepu, 2013). Despite this enduring research on students' understandings of the derivative, there is less research on the teaching of derivatives through using representations. The scarce research in this area focuses on the student outcomes of instructional treatments. Kendal and Stacy (2000) compared high school students' competency in derivative representations from two classes. Although both teachers used identical curricular materials, students from the two classes became proficient in different representations (graphical versus symbolic) based on their respective teachers' emphasis on one representation. The authors claimed that "students had different cognitive experiences and acquired different differentiation competencies which related directly to the ways they were taught" (p. 133). Similarly, Hähkiöniemi (2006) carried out a teaching period based on the APOS theory, during which he emphasized visual and symbolic representations. He agreed with Kendal and Stacy's (2000) claim, as he found that students could recognize derivative visually early on and calculate derivative at a point symbolically. Hähkiöniemi (2006) concluded that visual representations are a good fit for introducing and developing understanding of derivatives. Also using the APOS theory, Borji et al. (2018) designed a teaching intervention focused on graphical representations. While both the experiment and the control groups spent equal time on the concept of derivative, the students in the experiment group spent some time using a programming software with graphing facilities, and later performed better on a written tests on derivatives. In another study, instead of emphasizing visual and symbolic representations, Dwirahayu et al. (2017) prioritized physical representations, specifically analogies to the speed of a vehicle, to introduce derivatives to students. Compared to the control group, the authors showed that teaching with analogies resulted in higher student understanding of other representations of the derivative.

These interventions are promising in explicating the relationship between teaching and learning of derivatives using different representations. However, the research has yet to investigate how instructors, outside of interventions, think about their use of representations for teaching derivatives and how they design and sequence their instructional tasks based on their insights about student conceptions of derivatives.

Theoretical Framework

In this study, I use Zandieh's (2000) framework (Table 1) to: 1) ground instructors in both the knowledge at stake and students' conceptions of derivative; and 2) organize instructors' use of representations when proposing tasks for teaching derivatives with inquiry.

Table 1. Zandieh's (2000) adapted framework for the concepts of derivative.

		Representations			
		Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Difference Quotient)
Process-Object Layer	Ratio	Slope of the secant line	Average rate of change	Average velocity	Difference quotient
	Limit	Slope of the tangent line	Instantaneous rate of change	Instantaneous velocity	Limit of the difference quotient
	Function	Graph of the derivative function	Rate of change of a function	Velocity as a function of time	Derivative as a function

Zandieh (2000) organized students' conceptions by representation (graphical, verbal, physical, symbolic) and process-object layers (ratio, limit, function). The process-object layers are hierarchical, as each layer is found by taking the process of that layer over the previous layer as an object. For example, the limit layer is found by the *process of* finding the limit of the ratio as an *object*. Although not originally captured in the framework, Likwambe and Christiansen (2008) and Roundy et al. (2015) added two new representations: numerical and instrumental.

The numerical representation of the derivative corresponds to the ratio of change ($\frac{y_2-y_1}{x_2-x_1}$) is written with values, often with a table. The limit layer corresponds to when the denominator approaches zero; the function layer is presented as an array of numbers or set of ordered pairs of differences. The instrumental representation of the derivative is the rules of differentiation and lives only at the function layer.

Methods

This qualitative study uses data collected through interviews with eight college calculus instructors who were purposefully selected. To select the interviewees, I administered a survey, including the INQUIRE instrument (The INQUiry-Oriented Instructor REview; Shultz, 2020), to 48 college instructors who teach Calculus I with inquiry. I then used INQUIRE to purposefully select eight interviewees with different patterns of inquiry-oriented practices in the classroom. The final interview sample included eight participants (all pseudonyms): Justin, Adrian, Barry, Matthew, Gopher (he/him pronouns); Monica (she/her pronouns); Max (they/them pronouns); and Alex (all pronouns). All participants were tenured with more than four semesters experience teaching Calculus I with inquiry. I conducted four remote semi-structured interviews (1-2 hour long each; Figure 1) with each instructor, as they proposed up to eight tasks. Instructors could use their teaching materials to choose tasks they had used or to create new ones.

Using Zandieh's (2000) framework, I dedicated an interview to each representation. During each interview, the instructors were given two prompts and were asked to propose a task for each prompt that would transition students' conceptions from one layer (as pre-conception) to the next (as post-conception, see Figure 1). For example, as Prompt 1, participants were asked to propose a task situated within the physical representation of derivative that would help transition students' conceptions from the ratio to the limit layer (without the framework's language):

> Assume that you have already taught students about average velocity and want students to learn about instantaneous velocity at a point. Propose a task with the position of an object varying as a function of time, where students figure out the velocity of the object at a given moment.

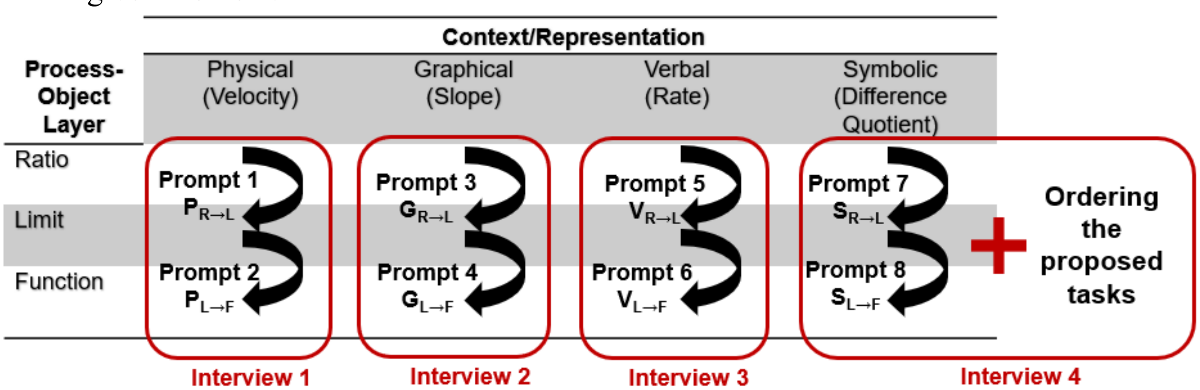

Figure 1. Structure of the interviews using Zandieh's (2000) framework. The code under a prompt, $X_{A\to B}$, gives: the representation X (P = Physical, G = Graphical, V = Verbal, and S = Symbolic), and the process-object layers A and B (R = Ratio, L = Limit, and F = Function). The arrow between the layers indicates the layer's precedence.

At the end of the final interview, I presented each instructor with all their proposed tasks and asked them to order them for introducing derivatives using inquiry. For the analysis reported here, I identified the representations and layers used within the proposed tasks without looking at the prompts' representation/layer code (those in Figure 1). I also coded representations of the derivative not in Zandieh's (2000) original framework. After coding, I listened to the interviews and read the transcripts to identify representations/layers that were mentioned but not written down in the task, as many instructors did not have enough time to write down all task details.

Findings

Due to space limitations, I present the representations used in the instructional task that instructors chose to use *first* when introducing derivatives to their students with inquiry. Because at the end of the final interview and after proposing eight tasks, the instructors re-arranged their proposed tasks in their preferred sequence for teaching derivatives, each instructor's *first task* is not necessarily the product of the *first prompt*. Figure 2 shows the prompts that generated each instructor's first task and the representations and layers identified in each. As their first task, the instructors chose to either use the task proposed for Prompt 1 (physical representation from the ratio to the limit layer) or Prompt 3 (graphical representation from the ratio to the limit layer). As shown in Figure 2's "Other reps/layers row", Gopher, Max, Barry, and Matthew reached out to representations and layers not mentioned in the prompts.

<table>
<tr><td>Prompt</td><td>Prompt(s) with reps/layers given</td><td colspan="4">Prompt 1
P: R → L</td><td colspan="2">Prompt 3
G: R → L</td><td>Prompts 1 and 3
P + G: R → L</td></tr>
<tr><td rowspan="2">First Task</td><td>Task's reps/layers used from prompt</td><td colspan="3">$P_{R/L}$</td><td rowspan="2">P_R</td><td rowspan="2">$G_{R/L}$</td><td>G_L</td><td>$P_{R/L} + G_{R/L}$</td></tr>
<tr><td>Other reps/layers</td><td></td><td>$G_{R/L}$</td><td>$N_{R/L} + S_L$</td><td>$G_F + S_{R/L}$</td><td>$V_{R/L} + S_{R/L}$</td></tr>
<tr><td></td><td>Participants</td><td>Alex
Monica</td><td>Gopher</td><td>Max</td><td>Adrian</td><td>Justin</td><td>Barry</td><td>Matthew</td></tr>
</table>

Figure 2. Derivative's representations and layers in the prompt that generated each instructor's first task, and representations and layers identified in their first tasks. Reps/layer is short for representations and layers. Each task has rep/layer codes similar to those in Figure 1. The task's layers are separated with slash (/) instead of arrows (→) here because I do not claim that the tasks necessarily transfer students' conception from one layer to another.

Alex, Monica, and Gopher provided similar tasks with tables of data for the position of a falling objects at various times and asked students to find the average velocity for various intervals. They then asked their students to find an estimate for the velocity at somewhere arbitrary between the existing data points. Gopher also used the graphical representation by defining secant and tangent lines, asking students to draw them on the curve of best fit for the data in Excel and finding connections between the lines' slopes and the velocities.

While Adrian and Max started their tasks similarly by providing the equation of the position of an object as a function of time, Adrian's task stayed within the ratio layer, as it asked students to only explain the physical meaning of some ratios (e.g., $\frac{R(5)-R(3)}{5-3}$). Max however provided multiple tables with times approaching $t = 5$ (e.g., 4.5, 4.75, 4.875) and asked students to complete the tables with average speed between the given times and $t = 5$. Max then asked students to estimate and write a formula for the object's speed at $t = 5$, thus using the limit layer of multiple representations (numerical and symbolic).

Looking at all representations used by instructors in their first task, everyone, except Justin, used the physical representation for introducing derivatives to their students, either encouraged by the prompt or by choice. Staying within the graphical representation only, Justin used an interactive GeoGebra applet to ask his students to find the slope of a secant line between two points on $f(x) = x^2$, as the distance between the points decreased. He then asked students to calculate the slope if Δx is zero, see what happens, and think about what the calculations reminded them of. Being cognizant of not using the physical representation, Justin explained that he does not use velocity "as the gateway application to move to calculus" because students get trapped in thinking that "a derivative is always a velocity rather than a velocity is always a derivative" (Interview 1).

Like Justin, Barry also chose the task he proposed for prompt 3 as his first task. Using a variety of functions and without mentioning secants, Barry asked students to find slope of tangent lines at various points. He then asked students to consider 2^x and plot $\frac{2^{x+h}-2^x}{h}$ in Mathematica with h getting smaller, ending the task by providing the limit definition of the derivative. As the only instructor to do so, Barry did not include the ratio layer in his task, but used the function layer.

Lastly, Matthew was a special case in the sense that he proposed the same task for both Prompt 1 and Prompt 3, saying that he wants students to "associate graphical representations to various parts of their work in the context of distance, velocity, time" early on (Interview 2). He also was the only instructor that used all four original representations (within ratio and limit layers) in his first task. Starting the task with the equation of the height of a bolt fired from a crossbow as a function of time, Matthew proposed a series of subtasks that asked students to: approximate the speed for a specific *t*, explain their approximation "using words only" (verbal), use the limit notation to write their findings, and identify secant and tangent lines on the graph of the function and find their slopes. Wanting students to become aware of the four main representations of the derivative, Matthew said that the task makes them "repeat the same set of questions over and over in different representations" (Interview 1).

Discussion

The findings show that the instructors in this study found graphical and physical representations, within ratio and limit layers, appropriate for introducing derivatives. These results align with Hähkiöniemi's (2006) and Dwirahayu et al.'s (2017) suggestions of using these representations early on. Nonetheless, half of the instructors also used other representations simultaneously. Moreover, while most instructor's use of the representations reflected the hierarchical order of layers in Zandieh's (2000) framework, one instructor (Barry) chose to not teach in that order. Further inquiry is needed to understand the ways in which calculus instructors utilize representations, separately and in connection to one another, at various conceptual levels to teach derivatives.

The study extends our understanding of inquiry teaching by attending to the content, as opposed to pedagogy, through their use of instructional tasks. As more mathematics faculty implement inquiry-based methods, it is essential that we understand their decision-making at the content level so that their specific needs can be met through curriculum or professional development initiatives. By attending to instructional tasks, we also gain a better understanding of what is actually offered to students during instruction and how it might impact their learning.

Acknowledgement

Funding for this work was provided by the University of Michigan's Rackham Graduate School to the author. I sincerely thank Dr. Vilma Mesa, Dr. Patricio Herbst, and the RTMUS lab for their continuous guidance and feedback.

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