Why Mathematicians ‘Fully Understand’: An Exploration of Influences on Their Understanding

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One of the primary mathematical practices of many professional research mathematicians is engaging with highly advanced mathematical concepts in their research. As mathematicians engage with these concepts in their work, they may or may not ‘fully understand’ them, which could include not being able to deduce why each step in a proof logically follows from the next. This preliminary study investigates what reasons professional mathematicians give for not ‘fully understanding’ mathematical concepts in their own work. One mathematician researching coding theory was interviewed, which utilized one of the participant’s own research journal articles as a launching point for discussion. Various reasons for the participant’s conceived amount of understanding were observed, including authoritative reasoning and academic pressures. Implications for future research on professional mathematicians are discussed.

Keywords: Advanced Mathematical Thinking, Professional Mathematician, Understanding

Introduction

Professional research mathematicians are constantly engaging with highly-abstract, advanced mathematical concepts in their academic endeavors, especially if they study pure mathematics. Nevertheless, even with the heavily abstract nature of the mathematics, mathematicians are successful at answering mathematical problems and publishing new research. Therefore, it is worth investigating how professional mathematicians engage with and understand highly-abstract, advanced mathematical concepts. Some research has investigated how mathematicians understand mathematical concepts (e.g., Flanagan, 2022; Shepherd & van de Sande, 2014; Sinclair & Gol Tabaghi, 2010; Soto-Johnson et al., 2016; Wilkerson-Jerde & Wilensky, 2011). Despite this, there is certainly a need to further analyze the intricacies of how mathematicians understand these concepts, and specifically what factors influence how mathematicians engage in this understanding. This preliminary research study aims to address this need.

Literature Review and Conceptual Framework

The study of mathematical understanding has been addressed in a variety of ways, including procedural versus conceptual understanding (Hiebert and Lefevre, 1986), intuitive, relational, symbolic, and logical understanding (Byers & Herscovics, 1977; Skemp, 1976, 1987), and the Pirie-Kieren dynamical theory of understanding centered on how one experiences growth in understanding mathematical concepts (Pirie & Kieren, 1989, 1994). Additionally, distinguishing between different types or levels of understanding has appeared in the research on proof, such as the distinction between semantic and syntactic proof productions (Weber & Alcock, 2004) and the need for strategic knowledge (Weber, 2001). Furthermore, research has demonstrated that professional mathematicians sometimes choose to not ‘fully understand’ all the mathematics they work closely with. Fully understanding a mathematical concept is impossible to define precisely, but a demonstration of a lack of full understanding could include choosing to use a theorem in a proof without knowing why the theorem is true, or accepting a proof is valid without being able to deduce why each step logically proceeds from the previous one.

In the context of proof-reading and validation, there is strong evidence that professional mathematicians will accept a proof as true or valid without using deductive reasoning (e.g., Weber et al., 2014; Weber et al., 2022). For example, Weber (2021) provided evidence that even set theorists, who work closely with the building blocks of mathematics, are sometimes content without being able to construct certain details of their derivations. In addition, professional mathematicians have demonstrated that they sometimes use authoritative reasoning when reading and validating proofs (e.g., Inglis & Mejía-Ramos, 2009; Mejía-Ramos & Weber, 2014; Weber, 2008; Weber & Mejía-Ramos, 2011). Harel and Sowder (1998) defined an authoritarian proof scheme to be when the source of an individual's truth claims come from a place such as a teacher or a textbook. Instead of teachers or textbooks, mathematicians sometimes appeal to the authority of reputable journals and famous mathematicians.

Research Questions

Two primary research questions guided this research study. The first question captures the broad purpose of the research, while the second pertains to the specific goals of this study.

1. (Broadly) How do mathematicians successfully work with and understand highly-abstract, advanced mathematical concepts?
2. (Specifically) What reasons do professional mathematicians give for why they believe they 'fully understand' or do not 'fully understand' the mathematical concepts they use in their own work?

Methodology

To address the above research questions, a one-hour, semi-structured Zoom interview was conducted with one professional research mathematician. This mathematician, Callie (pseudonym), was a female algebraist who specializes in coding theory. She was specifically chosen for this study because her area of research was tangentially related to my own mathematical expertise, which would allow for me to better understand the mathematics she discussed in the interview. This interview was video-recorded, and the audio was transcribed for data analysis. Furthermore, this interview consisted of three phases. Callie was first asked some basic questions to better understand her mathematical background. Afterwards, we discussed specific components of one of her own research journal publications. Questions were centered around aspects of the article where she referenced mathematics from other sources, to help assess whether she believed that she understood the mathematics she was using from other sources that she did not publish herself. The final phase consisted of questions directly pertaining to the research questions, two of which are provided below for reference.

1. Are there any times where you believe that you have encountered a new advanced or highly-abstract mathematical concept where you were able to successfully work with it, but did not 'truly understand' the concept?
2. Have there been any incidents where you read a research paper relevant to your own research where you utilized some of the mathematics in the paper, but did not understand why it was true?

The interview transcript data was subsequently coded in two phases. The initial phase of coding looked for general codes pertaining to the research questions. These initial codes were then refined and grouped together into more precise codes, which were applied to the transcript in a second phase of coding. After final adjustments were made, these codes were organized into four themes, which will be discussed in the following section.

Results

The results from the data analysis will be presented as a thematic analysis (Braun & Clarke, 2012) of the reasons that Callie provided for having certain levels of understanding for the mathematical concepts she uses in her own work. The coding process revealed four major themes, which are (1) authoritative reasoning, (2) external pressures, (3) academic influences, and (4) intellectual factors. Each of these themes will be addressed individually, with supporting evidence provided from the interview.

Authoritative Reasoning

One pattern that appeared frequently in the data was authoritative reasoning. This consists of when Callie stated that she believed or accepted something was true because it came from a trustworthy or authoritative source, such as a reputable journal. In the case of appealing to the authority of journal articles, Callie stated that "I also think that because of the peer review process, and all of that, typically I feel confident that results that have appeared in the literature are correct. You know sometimes that's based on journal reputation, right?" Moreover, in addition to the peer review process providing Callie with confidence in the validity of results, she expressed that if a result is important, then likely enough people have looked over the result to verify its correctness. Regarding a particularly important result in her field, she mentioned, "And so it's had lots and lots of eyes on it, so there's something that comes from people believing that this is true and knowing that other people have checked it…it's received a lot of attention." Another facet of authoritative reasoning came in the form of believing results were true because certain people were involved. When discussing an area of her own field of research where she did not understand all of the results, she stated that this is "an example of where we're looking to use something and we trust that the people who have done the work have done it correctly."

External Pressures

Another theme that came out of the interview data with Callie was external pressures. This theme consists of when her choice to try to understand a mathematical concept was directly influenced by external factors, such as time constraints due to academic duties. The most common type of external pressure was time constraints. Callie's busy schedule and its influence on her ability to take time to deeply understand all the mathematical concepts in her work was summed up when she stated:

> And you know now, there's just not the time to do that. I mean I have my own students I have to worry about. I also have an administrative role that's practically a full-time job, in addition to the faculty member role. And so, so that's part of it, it's the time.

Moreover, she emphasized that "it'd be hard to confirm every single thing that we use and actually create something new ourselves at the same time." In actuality, highly advanced mathematics can take time to understand, and there is a lot of it being produced. According to Callie, this makes it unrealistic to produce new results and 'fully understand' all related results.

Another type of external pressure is the natural influence of what the current research community believes is important. For example, Callie described how she and her colleagues were looking to find a sufficient condition for a mathematical result, even though they were not entirely sure what they were looking for. However, they did that because they "knew it would be well-received by the research community" if they could find it. Additionally, she mentioned that her research pursuits have shifted because she wants to "stay current." Specifically, she stated:

> The field has moved a bit, and so I kind of like to move a little bit too in order to stay current, you know, in terms of applications and things that people actually care about,

> like for funding agencies or journals...because I'm having to pivot. I shouldn't say having to; I'm choosing to pivot based on some internal motivation to follow external factors.

Even though these are reasons why a mathematician might choose to pursue certain research questions, they also demonstrate that there are academic pressures that influence what one chooses to focus on and try to better understand.

Lastly, Callie shared how her level of understanding on certain mathematical concepts has likely been affected by the availability of resources now versus when she was starting her career. In particular, now there are a whole host of relevant journal articles that we can access on the internet with minimal effort. However, twenty years ago, Callie described in contrast to having twenty-five online articles, "If I had gone to the library and I only have two papers related to a subject…I'm going to really pour into those in ways that I wouldn't do with the twenty-five." This comment is especially interesting because it highlights that the extent to which mathematicians choose to understand mathematical concepts for their own research may be changing due to technological advancements, which would need further research to verify.

Academic Influences

A third pattern that emerged from the data is academic influences. This consisted of when Callie discussed anything influenced how the actual mathematical results would be used or the nature of the mathematical content itself, including what field of study that the mathematical content came from. Callie explained how the field of study influences what she chooses to 'fully understand' by stating, "And in order to follow all that reason, I mean it might be a series of many many papers, you know, traversing other fields. So, it might be outside of what would be easily doable without help." This means that sometimes it may be difficult to understand the mathematics you are using, if you need to use advanced concepts that are outside of your area of expertise. Moreover, it is also possible that the field of research you engage in might influence the breadth of what you are able to understand, as suggested by Callie. She stated that "Different research areas are different as well…Coding theory's evolved a lot in the last twenty years. And so that means lots of different types of things go into my research program." She contrasted this with a mathematician who focused deeply on one idea for the entirety of his career, where he may deeply understand everything that he uses in his own work.

Another factor that appeared in the data was the influence of one's stage of career or current mathematical experience. Callie described how she believes she is "more comfortable now not knowing things than she once was," and questioned whether that is a "natural career progression" or not. Lastly, Callie expressed instances where the mathematical purpose of a concept for a journal article influenced how much effort she put into understanding it. Generally, she stated, "I use material from the literature and assume that it's correct, and I think that how far down that we go into that depends on what we're trying to do with the material." In the context of the journal article from the interview, she mentioned:

> You know there are times when you know, maybe you need to reference something and, you know, the results are there or you can see them, but you don't have full awareness of the supporting evidence. It certainly happens a lot, but in this case, you know we were trying to really nail down this value, and so I had spent some time with the techniques in that paper.

This sentiment was reoccurring in our discussion of her journal article, because she said that she understood the important references well but did not fully read or understand some of the references that were only provided for context or background.

Intellectual Factors

The final theme from the interview data was intellectual factors, which were any mentions of mental or intellectual influences on one's understanding, including Callie's mental capacity and how reasonable a mathematical statement appears. For instance, Callie expressed how the reasonability of a mathematical statement influences how much effort she needs to put into deeply understanding the mathematics.

> So sometimes my knowledge and awareness might come from hearing a talk about it and feeling like "Okay those techniques make sense; it seems reasonable; that's all I need." Other times, you know, it might be deeper confirmation if you feel like "Well that sounds surprising. I didn't think something like that could be true."

It is possible that when a mathematical statement appears reasonable, that this is influenced from other mathematics you know. In this case, one might believe that the statement is probably true based on other mathematical expertise or intuition. In addition to reasonability, Callie mentioned how she only has so much mental capacity to understand mathematical facts. She discussed this phenomenon by stating that most mathematicians cannot know everything in their field, and verbalized, "But to know the whole field is just not, at least for me, possible. I don't think it's possible for most people." These remarks provide support that mathematicians may not be able to 'fully understand' everything they work with, even if they would like to.

Discussion

The above thematic analysis provided supporting evidence that professional mathematicians do not always 'fully understand' the mathematical concepts they use in their work. Moreover, there are a plethora of reasons for why mathematicians have certain levels of understanding for these concepts. However, despite this, this study was preliminary in many ways and has some limitations. Most noticeably, this study only consisted of one mathematician researching coding theory. To further support these claims, it is important to research other mathematicians in similar ways, to see if Callie's experiences are shared among other mathematicians. A second limitation was that the chosen journal article was written over fifteen years before the interview. Due to this, some of Callie's remarks may have been misremembered, and she was not able to share as much detail about aspects of her producing the paper. In turn, much of the data came from the third phase of the interview. I believe that utilizing mathematicians' journal articles for interview data collection is valuable, but further data collection is certainly needed.

Moreover, the results from the analysis support the claims of the aforementioned literature that mathematicians appeal to authoritative reasoning and have reasons for not 'fully understanding' certain mathematical concepts they use (e.g., Inglis & Mejía-Ramos, 2009; Weber, 2008). In addition, this study contributed a concrete example of a mathematician who demonstrated numerous reasons for the understanding she had. However, further research is needed to delineate these various reasons why mathematicians do not 'fully understand' concepts. Is it mostly because of external and academic factors? Or is it also due to mathematical factors, such as using the concept in rich, mathematical "inventising" (Pirie & Kieren, 1994; see also Hadamard, 1945) or that the concept is too advanced or abstract for the individual? The above analysis provided some evidence that these could all be at work, but further research needs to be conducted. Lastly, the evidence in this study has implications for the teaching of advanced mathematics to students. If professional mathematicians are influenced in various ways on what they choose to 'fully understand,' we should not be surprised when students are similarly influenced. Instead, focus should be given to helping students learn to navigate what mathematical concepts should be 'fully understood' and what could be postponed until later.

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