University Students' Evaluations of Quantified Statements in Mathematics: A Survey Study

Derek Eckman
Arizona State University

Kyeong Hah Roh
Arizona State University

Erika David Parr
Rhodes College

Morgan Early Sellers
Colorado Mesa University

In this paper, we describe the results of administering a survey we created to investigate students' meanings for a universally quantified variable. Respondents with various levels of proof-course experience reviewed statement interpretations showcasing different meanings for quantified variables, indicating their preferred examples, which examples they believed to be viable, and a truth value for the statement. We developed the various interpretations from meanings described in our previous qualitative work on student quantification (e.g., Sellers et al., 2021). Our results suggest that students with a range of proof-course experience may benefit from reviewing diverse interpretations of a quantified statement. Still, students with less proof experience tended to accept more interpretations as viable or chose inappropriate truth values. In particular, we identify two non-normative interpretations most students believed were viable. We recommend proof-course instructors explicitly address these meanings in their classroom to foster student understanding of quantified statements.

Keywords: Quantification, proof education, scaling qualitative research, survey design

Introduction

Research has shown that students' interpretations of quantified mathematical statements play a crucial role in their understanding of advanced mathematical concepts in real analysis (Roh, 2010; Roh & Lee, 2017) and their proof writing and comprehension (Dawkins et al., 2021; Selden & Selden, 1995). Our previous work (e.g., Sellers, 2020; Sellers et al., 2018, 2021) has described five distinct ways students might interpret quantified variables. We have also shown that students' meanings for quantified variables can change from moment to moment, with the highest instability in meaning shown by calculus and transition-to-proof students (Sellers et al., 2017).

In this paper, we report the results of administering a survey we designed to (a) investigate the prevalence of these quantified variable meanings among collegiate students and (b) provide insight into instructional interventions that might promote students' appropriate quantification. In the survey, respondents with various levels of proof experience identified: (a) meanings they believed produced viable interpretations of a quantified statement, (b) which interpretations they preferred, and (c) a truth value for the statement. Our research questions emerged through reflection on what we might learn about students' quantification through a survey. Specifically, we investigated:

1. How does exposure to various interpretations of the quantified variable in a mathematical statement impact students' selection of the truth value for the statement? Do students' selections vary by proof-course experience?
2. Which meanings for quantified variables do students consider to be viable? Do students' selections vary by proof-course experience?
3. What meanings for quantified variables do students who select an inappropriate truth value prefer or believe to be viable after viewing the various interpretations?

Theoretical Framework

Sellers et al. (2018, 2021) propose the *Meanings for Quantified Variables* (MQ) framework to explain possible meanings students might have in a given moment for a quantified variable (see Table 1). We adopt the constructivist definition of *meaning*, grounded in the notion that individuals construct meanings in their minds through experience (Thompson et al., 2014).

Table 1: Categories of Meanings for Quantified Variables

Meaning	Mental Actions
MQ1	Checking if the predicate $P(x)$ holds for at least one value of x in X
MQ2	Checking if the predicate $P(x)$ holds for exactly one value of x in X
MQ3	Checking if the predicate $P(x)$ holds for all values of x in X
MQ4	Checking if the predicate $P(x)$ holds for spontaneously chosen values of x in X
NQ	No quantification (or absence of meaning for a quantified variable)

We describe how a student with each meaning might interpret a universally quantified statement, which we refer to as Statement 1 (S1): *For each real number x, let* $f(x) = x^4$*. Consider the following statement about the function* f*: For all real numbers x,* $\sqrt[4]{f(x)} = x$. The predicate $P(x)$ for S1 is $\sqrt[4]{f(x)} = x$, the domain of discourse X is the set of all real numbers $\mathbb{R}$, and the truth value of the statement is false. A student who uses MQ1 would check whether at least one real number satisfies the predicate $\sqrt[4]{f(x)} = x$. A student who uses MQ2 would check whether precisely one real number satisfies $\sqrt[4]{f(x)} = x$. A student who uses MQ3 would check whether all real numbers satisfy $\sqrt[4]{f(x)} = x$. A student who uses MQ4 would spontaneously choose real numbers and individually test whether each value satisfies $\sqrt[4]{f(x)} = x$. A student who exhibits no meaning for a quantified variable (i.e., NQ) might adopt an algebraic approach to investigating the predicate $\sqrt[4]{f(x)} = x$ rather than address how the quantified variable affects his interpretation of S1. A student normatively interpreting the quantified variable x in S1 would likely conclude that S1 is false by using MQ3 to identify a negative real number as a counterexample. However, other students might determine that S1 is false using MQ2, NQ, or MQ4, arriving at a normative truth value using an inappropriate quantification.

Survey Design and Methodology

Our survey instrument contained one universally and one existentially quantified statement. In this paper, we focus on S1, the universal statement. For S1, we included seven interpretations for x based on the MQ framework (see Table 2): one interpretation for each of the five meanings, and a second interpretation for both MQ1 and MQ3. In the first MQ1 and MQ3 examples, we included only values of x that satisfy the predicate $\sqrt[4]{f(x)} = x$ (which we called "E" for "examples"). In the second MQ1 and MQ3 interpretations, we included only values of x for which the predicate fails (which we called "CE" for "counterexamples").

For both quantified statements in the survey, the respondents answered questions in four distinct phases (see Figure 1). In Phase I, the students viewed and selected a truth value for the statement (options included *true, false*, *cannot be determined*, and *other*). The students then read a definition of "viable:" *capable of working, functioning, or developing adequately* (Merriam-Webster, n.d.). We included this definition to minimize the differences in how students might understand the term "viable" in our survey prompts.

Table 2: Text of the Hypothetical Interpretations for S1 on the Survey

Meaning	**Text of hypothetical interpretation for S1 on survey**
NQ	I would just plug in the function formula to the equation in the statement. The statement means that x is a real number and $\sqrt[4]{f(x)} = \sqrt[4]{x^4} = x$.
MQ1-E	I would expect to see there is at least one real number for x so that $\sqrt[4]{f(x)} = x$. If $x = 2$, then $\sqrt[4]{f(2)} = \sqrt[4]{16} = 2$. So, I have at least one real number for x so that $\sqrt[4]{f(x)} = x$.
MQ2	I would expect to see there is only one real number for x so that $\sqrt[4]{f(x)} = x$. But I found more than one real number for x so that $\sqrt[4]{f(x)} = x$. If $x = 2$, then $\sqrt[4]{f(2)} = \sqrt[4]{16} = 2$. If $x = 3$, then $\sqrt[4]{f(3)} = \sqrt[4]{81} = 3$.
MQ3-E	I would expect to see no matter what number I select for x, if x is a real number, then $\sqrt[4]{f(x)} = x$. If $x = 2$, then $\sqrt[4]{f(2)} = \sqrt[4]{16} = 2$. If $x = 3$, then $\sqrt[4]{f(3)} = \sqrt[4]{81} = 3$. I imagine that if I were to have unlimited time, I could keep selecting a different real number for x each time and see that I would have $\sqrt[4]{f(x)} = x$.
MQ1-CE	I would expect to see that there is at least one real number for x so that $\sqrt[4]{f(x)} = x$. If $x = -2$, then $\sqrt[4]{f(-2)} \neq -2$ because $\sqrt[4]{f(-2)} = \sqrt[4]{16} = 2$. Well, I will keep trying to check other real numbers to see whether I have at least one number for x so that if x is a real number, then $\sqrt[4]{f(x)} = x$.
MQ3-CE (Normative)	If I choose a real number -2 for x, $\sqrt[4]{f(x)} \neq x$ because $\sqrt[4]{f(-2)} = \sqrt[4]{16} = 2$. So there is at least one real number x such that $\sqrt[4]{f(x)} \neq x$.
MQ4-E/CE	It could be $\sqrt[4]{f(x)} = x$ or it could be $\sqrt[4]{f(x)} \neq x$. For instance, if $x = 2$, $\sqrt[4]{f(x)} = x$ because $\sqrt[4]{2^4} = 2$. On the other hand, if $x = -2$, $\sqrt[4]{f(x)} \neq x$ because $\sqrt[4]{(-2)^4} = 2$.

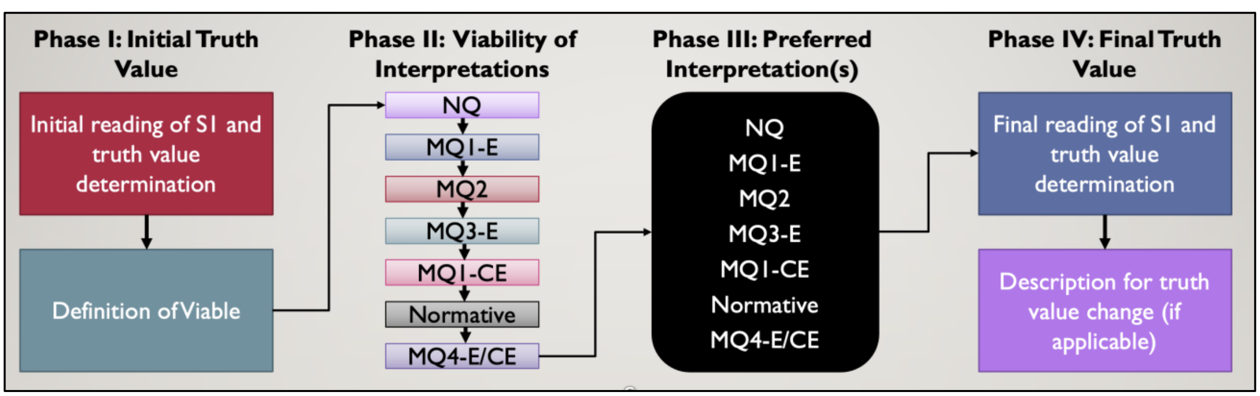

Figure 1: Visual Depiction of the Four Phases of the Survey for Statement 1

In Phase II, students read individual interpretations of the quantified statement, indicating whether they believed each example to be viable. In Phase III, students who indicated at least one interpretation was viable (a) reviewed their previous viability choices and (b) selected one or more preferred interpretations to reason about the truth value of S1. In contrast, students who claimed that no examples were viable described an alternative interpretation they considered viable. In the final phase, Phase IV, students again read the quantified statement and assigned a truth value. Students whose final truth value for S1 differed from their original truth value were prompted to explain why their choice changed.

We collected responses from 108 students enrolled in calculus courses and a proof-based geometry course at two universities in the United States. After comparing students' overall survey completion time, item completion times, and truth-value-change justifications, we deleted eight responses that appeared to be unreliable. The final sample of 100 students included 37 respondents who self-reported no proof-based course experience, 39 who reported taking one proof course, and 24 who reported participating in more than one proof course.

Results

In this section, we describe the results of our preliminary descriptive analysis of the data in relation to our three research questions regarding (a) the impact of the interpretations on students' truth value selections, (b) students' beliefs regarding the viability of the various hypothetical examples, and (c) which interpretations students who persisted in claiming S1 is true preferred or believed to be viable. We devote a subsection to each research question.

RQ1: Impact of Hypothetical Interpretations on Students' Truth Value Selections

We found a significant difference in the distribution of students' chosen truth values before and after reviewing the hypothetical interpretations, which we show in Table 3. Specifically, 69% of the 100 students initially indicated S1 is true, and 70% of the overall sample concluded that S1 is false after viewing the interpretations. This change in students' perception of S1's truth value seems to have emerged from their reflection on our survey items. For example, two students who changed their truth value from "true" to "false" explained that "I viewed instances where this statement couldn't be true" and "I did not initially think…negative numbers would not be viable solutions, as presented by multiple students."

Table 3: Initial and Final Truth Value Selections for S1 (Overall Sample of 100 Students)

Initial Chosen Truth Value	**The statement is true**	**69%**
	The statement is false	24%
	We cannot determine if the statement is true or false	6%
	Other (Please explain)	1%
Final Chosen Truth Value	The statement is true	26%
	The statement is false	**70%**
	We cannot determine if the statement is true or false	3%
	Other (Please explain)	1%

**Highlighted rows indicate the most selected truth value among respondents*

When we separated students' responses by proof-course experience, we discovered that students with little or no proof experience initially selected S1 is true in much greater numbers than those with more than one proof-based course (see Table 4). We also noted a large gain in the percentage of students who indicated that S1 is false after reviewing our interpretations (~50 percentage point increase for each group). Still, due to the near-universal initial response that S1 is true by students with no proof experience, we found that over 40% of calculus respondents still maintained that S1 is true after reviewing all seven interpretations.

Table 4: Initial and Final Truth Value Selections for S1 (100 Students Sorted by Proof-course Experience)

		0 Proof Courses ($n = 37$)	1 Proof Course ($n = 39$)	More than 1 Proof Course ($n = 24$)
Initial Chosen Truth Value	**True**	**34 (92%)**	**26 (67%)**	9 (38%)
	False	3 (8%)	10 (26%)	**11 (46%)**
Final Chosen Truth Value	**True**	15 (41%)	6 (15%)	5 (21%)
	False	**21 (57%)**	**30 (77%)**	**19 (79%)**

**Highlighted cells indicate the most commonly selected truth value among respondents. Students who selected "Cannot be Determined" or "Other" for their initial or final truth value are not included in this table.*

RQ2: Students' Determination of Viability for Individual Interpretations

We chose the order in which the hypothetical interpretations were presented to students strategically, considering that some students may not initially consider negative real numbers for values of x. We first presented an algebraic example, followed by arguments utilizing positive real numbers for x, and concluded with interpretations using negative real number values for x.

The following table, Table 5, shows the percentage of students in each proof-experience category that indicated a particular interpretation is viable. We found that at least 50% of students with no proof experience accepted 6 of the 7 interpretations as viable. Although students with proof experience tended to select fewer interpretations as viable, we noticed that a majority of students with proof experience still believed NQ, MQ3-CE (the normative interpretation), and MQ4 examples were viable. In contrast, the percentage of students who accepted the MQ1-E, MQ2, and MQ3-E interpretations decreased for each proof experience category. Our data imply that for our respondents, proof course experiences may have impacted their viability beliefs about some interpretations (i.e., MQ1-E, MQ2, MQ3-E) but not fully convinced them that other interpretations are unviable (i.e., NQ, MQ4).

Table 5: Viability Selections for S1 (100 Students Sorted by Proof-based Course Experience)

Proof Experience	**NQ**	**MQ1-E**	**MQ2**	**MQ3-E**	**MQ1-CE**	**MQ3-CE (Norm)**	**MQ4-E/CE**
0 courses (n = 37)	**27 (73%)**	**23 (62%)**	**20 (54%)**	**30 (81%)**	13 (35%)	**24 (65%)**	**20 (54%)**
1 course (n = 39)	**22 (56%)**	12 (31%)	10 (26%)	**24 (62%)**	16 (41%)	**29 (74%)**	**26 (67%)**
More than 1 course (n = 24)	**13 (54%)**	7 (29%)	4 (17%)	9 (38%)	9 (38%)	**20 (83%)**	**14 (58%)**

**Interpretations highlighted indicate more than 50% of respondents selected "viable"*

RQ3: The Preferred and Viable Interpretations of Students who Selected S1 is True

A vast majority of students in our sample concluded that S1 is false after viewing various hypothetical interpretations we provided in our survey. Still, a non-trivial minority of students

(26%) continued to insist that S1 is true (see Tables 3, 4). In this section, we examine the preferred and viable interpretations indicated by these students.

The following table, Table 6, contains two rows. The first row shows the percentage of the 26 students who insisted S1 is true that selected a particular meaning among their preferred interpretations. The second row displays the percentage of the 26 students who selected "true" that indicated an interpretation was viable. From these two rows, it appears that students who insisted that S1 is true preferred examples that only utilized positive real number values for x (i.e., MQ1-E, MQ3-E).[1] We also found that many students who indicated that S1 is true rejected interpretations that used negative values of x. For instance, no student who claimed S1's final truth value is true preferred the normative counterexample-based interpretation. Additionally, only 23% of these students claimed the normative interpretation is viable (whereas 91% of the students who claimed S1 is false stated the normative example is viable).

Table 6: Preferred and Viable Interpretations of the 26 Students whose Final Truth Value for S1 was "True"

	NQ	**MQ1-E**	**MQ2**	**MQ3-E**	**MQ1-CE**	**MQ3-CE (Norm)**	**MQ4-E/CE**
Students preferring interpretation ($n = 26$)	**12 (46%)**	**6 (23%)**	2 (8%)	**8 (31%)**	1 (4%)		3 (12%)
Students indicating interpretation is viable ($n = 26$)	**18 (69%)**	**14 (54%)**	11 (42%)	**20 (77%)**	7 (27%)	6 (23%)	9 (35%)

**Interpretations highlighted indicate the 2-3 highest (or lowest) percentages in each row*

Discussion and Conclusion

Our study aimed to gain insight into meanings for quantified variables that students preferred or believed to be viable for interpreting statements. Our survey constitutes an attempt to study these meanings, which emerged from our previous qualitative work (e.g., Sellers et al., 2017), on a larger scale. Our survey varied from our clinical interviews (Clement, 2000) in two distinct ways. First, in the survey, we gave students various statement interpretations and asked them to indicate which examples they believed were viable, while in the interviews, we modeled students' naturalistic interpretations of statements. Second, in the survey, we asked students to denote one or more provided arguments they preferred to use for interpreting the statement, while in the interviews, we assumed students' expressed ideas were their preferred interpretation.

Our first research question was related to how students' reading of various interpretations might impact their truth value choice for S1. Our results show that students selected an appropriate truth value for S1 at much higher rates after viewing the interpretations, although 41% of calculus respondents still maintained that S1 is true after reviewing our examples. This finding indicates that for many calculus students, pondering various examples of statement interpretations may not be sufficient to develop normative meanings for quantified statements.

Our second research question was related to the meanings students believed to produce viable interpretations of S1. Our results show that students with no proof experience might accept a broad range of non-normative meanings for the variable x. Additionally, our findings indicate that many students (regardless of proof experience) believe that NQ and MQ4 are viable

[1] We do not include the NQ example because 61% of students who selected S1 is false believed it to be viable.

quantifications. However, students' use of NQ or MQ4 is problematic because they might utilize these non-normative meanings to provide the appropriate truth value of S1. We thus encourage calculus and proof-based course instructors to facilitate student discussion activities to compare NQ, MQ4, and normative interpretations of quantified statements in their curricula.

Our third research question was related to students who maintained S1 is true after reviewing various interpretations of the statement. Our results show that these students preferred algebraic examples or those that only utilize positive real number values of x. These students' preference for NQ was unsurprising (to us), since we have previously reported that calculus students often exhibit NQ while interpreting statements (Sellers et al., 2017). Our finding that many students who maintained that S1 is true also believed that MQ1-E and MQ3-E are viable is a novel contribution to the quantification literature. We conjecture that these students might have relied more on examples than quantification to interpret S1. In this case, we would consider these students' rejection of MQ3-CE and MQ1-CE to be a natural consequence of their focus on interpreting S1 through examples. Specifically, these students may have believed that values of x that did not satisfy the predicate $f(x) = \sqrt[4]{x}$ were irrelevant to the interpretation process because they constituted "non-examples" of S1.

In summary, our survey results both verify and extend our previous qualitative findings. For instance, Sellers et al. (2021) described the existence of the five meanings in their MQ framework. In this paper, we extend these findings by showing (a) which meanings students prefer or believe are viable to interpret a universally quantified statement and (b) patterns in the meanings of students who believe a false universally quantified statement is true. We also provide insight into how teachers might implement Sellers et al.'s (2021) suggestion that instructors construct activities to address students' non-normative meanings for quantified variables. Our findings highlight one possible activity: asking students to compare NQ, MQ4, and normative interpretations for a universally quantified variable in a mathematical statement. In future studies, we plan to investigate further the prevalence of non-normative quantification among students and potential instructional interventions to facilitate students' construction of mathematical logic.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Lawrence Erlbaum.

Dawkins, P. C., Roh, K. H., Eckman, D., & Cho, Y. K. (2021). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 518–527).

Roh, K. H. (2010). An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics, 73*(3), 263–279. https://doi.org/10.1007/s10649-009-9210-4

Roh, K. H., & Lee, Y. H. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education, 3*(1), 34–68. https://doi.org/10.1007/s40753-016-0039-9

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123–151. https://doi.org/10.1007/BF01274210

Sellers, M. (2020). *Students' quantifications, interpretations, and negations of complex mathematical statements from calculus* [Unpublished doctoral dissertation]. Arizona State University.

Sellers, M. E., Roh, K. H., & David, E. J. (2017). A comparison of calculus, transition to proof, and advanced calculus student quantifications in complex mathematical statements. *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, 285–297.

Sellers, M. E., Roh, K. H., & David, E. J. (2018). Various meanings a student uses for quantified variables in calculus statements: The case of Zack. *Proceedings of the 40th Annual Meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education*, 580–587.

Sellers, M. E., Roh, K. H., & Parr, E. D. (2021). Student quantifications as meanings for quantified variables in complex mathematical statements. *The Journal of Mathematical Behavior*, *61*, 1–16. https://doi.org/10.1016/j.jmathb.2020.100802

Thompson, P., Carlson, M., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (pp. 1–24). University of Wyoming. http://pat-thompson.net/PDFversions/2013MagsInAlg.pdf