APOS Levels, Stages, and Coordinated Topics as Transitional Points
Case of Range of a Linear Transformation

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This paper discusses findings from an ongoing study investigating mental mechanisms involved in the conceptualization of linear transformations from the perspective of APOS theory. Data reported in this paper came from 44-first year linear algebra students' responses on a task regarding the range of a linear transformation. Our analysis revealed differences in APOS Levels, Stages, and Coordinated topics. Moreover, our findings pointed to connections between Levels/Stages of the range concept and mental mechanisms of representations of matrix multiplications. That is, our findings revealed that the absence of these representations from ones conceptualization process may result in irreversible misconceptions. More importantly, we identified more transitions occurring among Levels/Stages when linear combination expressions were considered as representations of matrix multiplications.

Keywords: Linear Transformations, APOS theory, Mental Mechanisms, and Transitional Points.

In linear algebra education, many topics, such as vector space and eigen-theory, have been studied extensively. To this point, linear transformations received very little attention. In the Literature, we found three studies that investigated connections between transformations and function ideas (Andrews-Larson, et al. 2017; Zandieh et al. 2017; Turgut, 2019), and only one by Oktac, et al. (2021) that investigated mental mechanisms of the null space of a linear transformation from the perspective of APOS theory. Our ongoing work is too investigating various features of linear transformations from the perspective of APOS theory. In this paper, we discuss APOS Stages, Levels and Coordinated topics our participants displayed in their responses to a question regarding the range concept. Our main goal was to examine potential transitional points in the learning of the range concept.

Literature and Theoretical Perspective

The Literature on Linear Algebra Education

Two recent publications, the ICME-13 Monographs and Journal of ZDM 2019 special issue on linear algebra, compiled most up-to-date research on linear algebra education. Between the two, some studies investigated general topics such as instructional approaches (Trigueros, 2018) and visualization (Harel, 2019) and others looked at specific topics such as vector spaces (Caglayan, 2019), eigen-theory (Karakok, 2019), and linear independence (Dogan, et al., 2022; Dogan, 2019). Many of them included, in their analysis, theoretical perspectives; for example APOS (Oktac, 2021), Tall's three worlds of mathematical thinking (Stewart et al, 2019), and Harel's DNR concepts (Harel, 2018).

Up to this point, we found only four studies concerning linear transformations. Three of which are by Andrews-Larson, et. al. (2017), Zandieh, et. al. (2017), and Turgut (2019). The forth is by Oktac, et. al. (2021). The first three studied the conceptualization of matrices as linear transformations. These authors discussed how the knowledge of the function concept had implications for one's identification of linear transformations. The fourth study by Oktac et. al. (2021), borrowing from APOS theory, discussed APOS Levels as displayed in their student responses to a question regarding the null space of a linear transformation, specifically, focusing on the pre-image notion. Their findings pointed to the role of reversal of linear transformations in the conceptualization of pre-image and null space ideas. Our work reported here, also borrowing from APOS, studied our participant responses to a question concerning the range of a linear transformation.

Theoretical Perspective-APOS

APOS theory describes features of mental mechanisms applied in the formation of the knowledge of advanced mathematical topics. The theory describes the mental mechanisms through Stages, namely, Action (A), Process (P), Object (O), and Schema (S). In our analysis, we considered APOS Stages as mental mechanisms our participants were working with at the time of the data collection. The theory considers the Stages (A, P, O, S), interchangeably, to stand for conceptions, structures, mechanisms, and stages (Arnon et. al., 2014).

According to the theory, learners are functioning at an Action Stage if they are acting on entities via external stimuli. For example, one may observe a learner functioning at an Action Stage if the learner feels the need to solve the system Ax=0 before determining the linear independence of a set of vectors. Here, solving the system Ax=0 provides an external stimulus for the learner.

APOS considers a learner to be working at a Process Stage if the learner determines the linear independence through a mental recognition (an internal stimulus) of certain features of vectors such as the ones with the same or different component values. Here, the mental recognition of the features provides an internal stimulus for the learner.

According to Oktac, et. al. (2021), “When Processes are encapsulated they become Objects to which Actions or Processes related to other concepts can be applied.” (p. 3). Thus, a learner identifying linear independence via the comparison of number of vectors and the dimension of a space is considered to have encapsulated Processes. That is, the learner is considered to be acting at an Object Stage, having already objectified vectors as objects of a space (set).

According to APOS, there are additional conceptual entities that can facilitate the development of and transitions between Stages. The ones that are relevant to our work are named as Coordination, Reversal and Levels. Coordination is described as involving mental mechanisms where two or more mechanisms united to give rise to a new Stage. Reversal is described as mental processes being reverted. Levels are defined as partial mental mechanisms whose features fall in-between Stages. An example of a Level, between Process and Object Stages, is when a learner considers the linear independence concept in the context of a set of three vectors with only one having a zero component value. Here, the learner is clearly functioning with an internal stimulus, mentally recognizing a pattern among component values of vectors. That is, the learner is applying a mental mechanism acting at a Level between Process and Object Stages.

APOS, furthermore, considers Stages as invariant constructs and asserts the necessity of each Stage for the development of successive Stages (Arnon et. al. 2014).That is, Stages are considered to be sequential. Levels, on the other hand, are considered non-invariant and may vary. The theory further elaborates that “a subject may be able to move to the next Level or Stage rapidly so that a Level is skipped, done very quickly, or is not observable in the already acquired higher Level or Stage” (Arnon et. al. 2014, p. 139).

Methodology

Our data came from 44 first-year matrix algebra students' responses to a question on the nullspace and range of a linear transformation (see fig.1) at a US institution from spring 2020 semester. Student came from a matrix algebra course met twice a week at noon via Blackboard, an online course delivery platform, for an hour and 20 minutes. The question was administered before the end of a class meeting through a problem solving session lasting about 20 minutes. Groups of three to four were formed and sent to designated online rooms for groups to discuss the question among themselves. Instructor was absent from the rooms and discussions. Before leaving the main class meeting room to their designated rooms, students were instructed to, individually, submit online, by 11:59 pm the same day, their own responses with explanations in detail. The particular class meeting ended by groups finishing their discussions in the designated rooms. The student population comprised of engineering, science, and mathematics majors. Linear algebra was the first theory-oriented course with high abstract level that the students took. The students came with very little or no maturity for the high abstraction required by this course.

We analyzed student responses to the question from the perspective of APOS theory. Specifically, we, first, categorized the responses based on student descriptions of the range concept (or its vectors) of the transformation given in Figure 1. Next, we identified APOS Stages and Levels displayed in responses of each category. In this paper, we discuss only the findings that came from part ii of the question seen in Figure 1.

Consider T:R^2->R^2 with $T([a\ b\ ]) = [1\ 1\ 1\ 1\ ][a\ b\ ]$. State at least three vectors that are:
i) In Null space (T) ii) In Range (T). Explain your reasoning in detail.

Figure 1. Mathematical task considered during group discussions and individual responses submitted online.

Results

We identified five distinct categories. Table 1 summarizes categories, Levels, Stages, and Coordinated topics. Overall, categories 1 and 2 had the highest tally of responses (12). The highest variability among Levels/Stages came in category 3 with three different mental mechanisms and the least variability came in category 5 with one Stage. Each of the categories 1-3 considered a representation of the matrix multiplication given in Figure 1. By means of these representations, some responses in these categories applied external stimuli in determining range vectors. One external stimulus was the evaluation of the matrix multiplication, A$[a\ b\ ]$, mainly in category 1; the general form, $[a\ +\ b\ a\ +\ b\ ]$, in category 2; and the linear combination form, $a[1\ 1\ ] +\ b[1\ 1\ ]$, in category 3. Each of these evaluations were done for specific values of *a* and *b*. The values of *a* and *b* took varying roles (for the participants) from *(a, b)* pairs being vectors of the domain of the transformation to *a*, *b* values being scalars for the linear combination form, $a[1\ 1\ ] +\ b[1\ 1\ ]$.

Mental mechanisms applying internal stimuli were also present, in these three categories, in an increasing trend from category 1 to 3. That is, category 1 (discussed in detail below) included about 8% of responses (1 out of 12 responses in the category) with the application of internal stimuli. This presented itself when students considered a general structure, out of the matrix multiplication given in Figure 1, for range vectors making statements similar to: "the multiplication of the matrix and *a, b* values gives top and bottom the same." Category 2 included about 58% of responses (7 out of 12 responses in the category) where the expression $[a\ +\ b\ a\ +\ b\ ]$ was interpreted as range vectors having the same component values without any external calculations. Category 3 included about 70% of responses (7 of 10 responses in the category) mentally connecting the linear combination form, $a[1\ 1\,] + b[1\ 1\,]$, to a spanning set or a basis set of the range, and considering range vectors as the object of these sets.

In comparison to those in categories 1-3, category 4 (total 6) displayed drastically different mental mechanisms. Responses in this category determined range vectors based upon the structure of vectors of the null space. They did this, for the most part, acting on internal stimuli. This category is discussed in detail later in the paper.

Responses (total 4) in category 5 were either incomplete or irrelevant to the task on hand. One response, for instance, gave 2x2 matrices as range vectors. Three of them calculated externally the reduce row echelon form (Johnson et. al. 2002) of amalgam of the matrix in Figure 1 with specific values of *a* and *b*. They, however, fell short in, correctly, identifying range vectors by using the rref form. Two of these left their work at the rref form without any further explanations. The third one made a statement along the lines of: “no range since the rref implies no solution.” In class, the rref process was introduced to identify vectors of a basis/a spanning set for the range of linear transformations. The three students appeared to have recognized the rref process having something to do with linear transformations but fell short in displaying any knowledge of connections between the rref forms and bases/spanning sets.

Let’s us now look at categories 1 and 4 with little more details in order to give readers further insights into the nature of responses and mental mechanisms.

Table 1. Category, Level, Stage, Coordinated Topic, and Tally.

Range	Tally	#/Level/Stage	Coordinated Topics
1.MM*	12	2/A-P*	MM/Pi*/NS*
2. CV*	12	2/A-P	MM/Pi.
3. LC*	10	3/A-O*	LC/B*/S*
4. NS*	6	2/P-O	NS
5. IR*	4	1/A*	RREF/SS*

**A-Action Stage; P-Process Stage; O-Object Stage; *on prior-knowledge (not on Range); MM-Matrix Multiplication; Pi-Pre-image; NS-Null Space; LC-Linear Combination; S-Spanning set; B-Basis; SS-Solution set; CV-Component values; IR-irrelevant.*

Category 1: Range as the Result of Matrix Multiplications

There are 12 responses in this category. In order to determine range vectors, majority of participants evaluated matrix multiplications for specific values of *(a, b)* pairs, which represented (for the participants) domain vectors of the transformation. Moreover, some considered any values for *a* and *b* while others considered values with the condition, *a=b*. The response in Figure 2 exemplifies the latter behavior. This response is carrying out the multiplication of the matrix in fig 1 with *a, b* values where *a=b* i.e $[2\ 2\,]$. Figure 3 gives a case for the former behavior. This response is using different values for the pair *(a, b)*. For example, the participant is multiplying the pair $[10\ 7\,]$. with the matrix in fig 1 and obtaining a range vector $[17\ 17\,]$. Mental mechanisms displayed in both responses in Figures 2 and 3 needed external stimuli in determining range vectors. Thus, the mental mechanisms were acting at Action Stages.

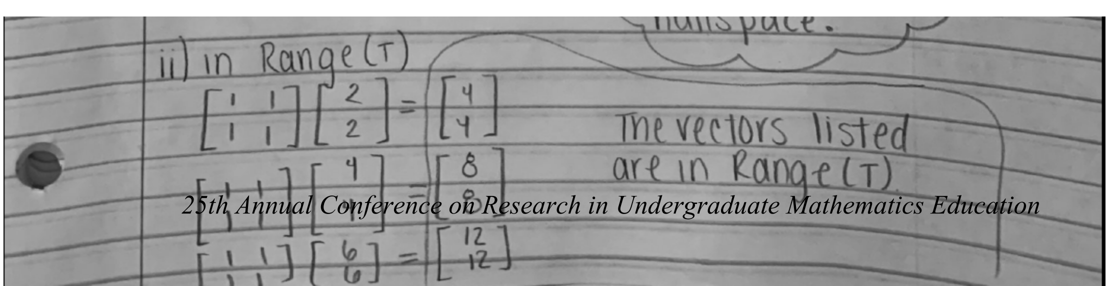

Figure 2. Domain vectors with a=b -externally calculating matrix multiplication.

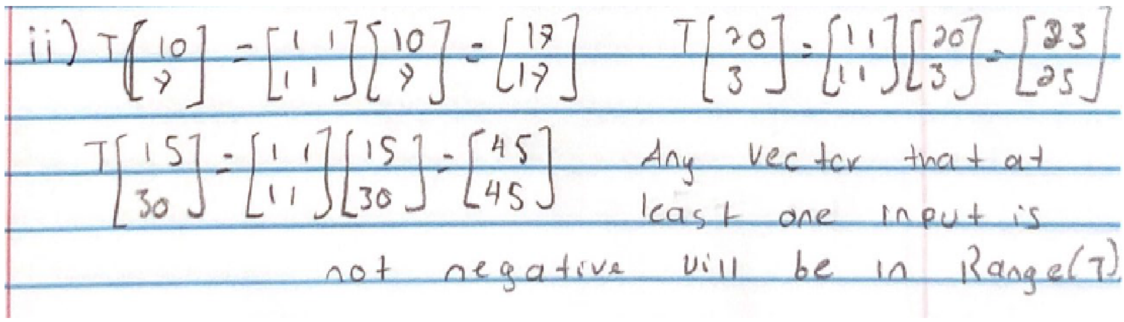

Figure 3. Domain vectors with any a, b values -externally calculating matrix multiplication.

In addition to the two behaviors described above, there were also a few responses revealing mental mechanisms acting at Process Stages. That is, they revealed participants considering matrix multiplication processes internally describing what would be obtained if a multiplication was carried out without any external computations. Figure 4 portrays one such tendency. This participant is considering generic pairs *(a, b)* and asserting that "all these vectors are the answer of $T([a\ b\ ]) = [1\ 1\ 1\ 1\ ][a\ b\ ]$" with no external calculations. The mental mechanism revealed in this response is clearly the one acting at a Level closer to a Process Stage.

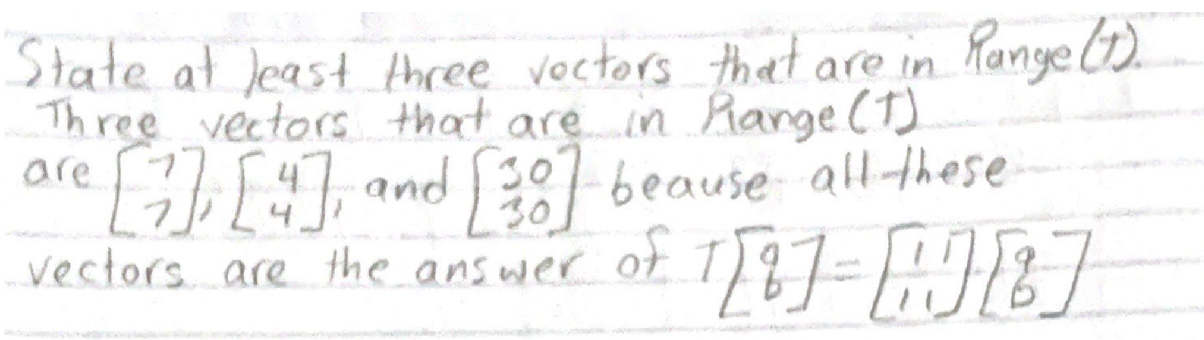

Figure 4. Consideration of matrix multiplication processes internally (mentally).

Category 4: Range Compared to Null Space

Responses in category 4 differed noticeably from those of categories 1-3. As far as we know, this category reveals mental mechanisms that have not yet been reported in the Literature. There are 6 responses in the category. This category determined range vectors based upon the structure of vectors of the null space of the transformation in Figure 1. We identified two distinct mental mechanisms in this category. One considered range vectors as those (from the domain of the transformation) with characteristics dissimilar to the ones in the null space of the transformation. Response in Figure 5 is displaying this tendency. This participant is considering vectors with same component values as vectors of the range. The participant's comparison of range vectors to vectors of the null space is clear in the explanation, “product with [1 1 1 1] will not give a [0 0] matrix.” Mental mechanisms displayed in similar responses are classified as acting at Levels closer to Process stages.

There were also responses considering the range and null spaces as complementary sets making up the domain of the transformation. Figure 6 is revealing this behaviour clearly. This participant is declaring that any vector that is not in the form $[x \ \ -x]$ (identified as the structure of vectors of the null space) as the vectors of the range. In light of the following statement “any vector of the form [x;-x] <u>exists</u> within the null space of T,” (Emphasis is on the term “exists”) given for part i of the question, one can identify mental mechanisms, displayed in this response, as the ones acting at Levels closer to Object Stages. That is, in this response, vectors are considered as objects of the null space, subsequently objects of the range set.

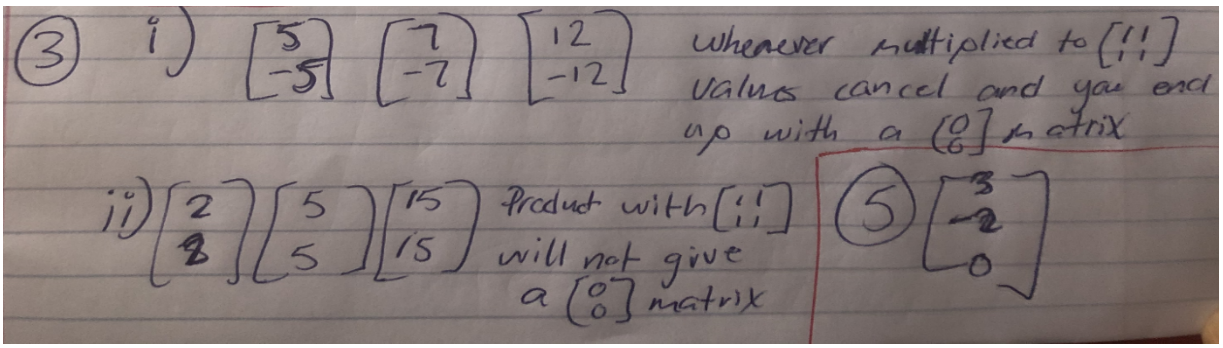

Figure 5. Range vectors in comparison to the structure of vectors of the nullspace.

3i.) Three vectors in the Nullspace of T are [1;-1], [2;-2], and [-3;3]. In fact, any vector of the form [x;-x] exists within the nullspace of T.
3ii.) Three vectors in the range of T are any vectors NOT of the form [x;-x], which creates an infinite list. Some examples are [1;-4], [5;2], and [x; y] such that y is not equal to x.

Figure 6. Vectors other than those in the nulspace (Response was typed by the student).

Unfortunately, even though mental mechanisms applied in the responses of this category were acting at higher Levels/Stages, inaccurate knowledge structures were displayed. This seemed to have been the result of, somehow, participants’ attention being directed to the null space concept, escaping any ideas from categories 1-3 (matrix multiplication representations).

Discussion and Conclusion

In this paper, we discussed findings from an ongoing study investigating mental mechanisms involved in the conceptualization of linear transformations from the perspective of APOS theory, specifically, findings from responses of 44-first year linear algebra students on a task concerning the range of the linear transformation given Figure 1. Our main goal was to identify transitional points in the learning of range ideas.

Overall, our findings show that responses employing various representations of the matrix multiplication, A$[a \ b]$, in Figure 1, presented evolving Levels of mental mechanisms. That is, from category 1 to category 3, we documented mental mechanisms acting at evolving Levels from Action Stages to Object stages.

The highest variability among Levels occurred in category 3 when participants considered linear combination forms, $a[1\ 1\ ] +\ b[1\ 1\ ]$, in place of matrix multiplications. In this category, we documented mental mechanisms acting at all three APOS Stages, Action, Process and Object. Object Stage mental constructs were absent from categories 1 and 2. Indeed, category 3 and category 4 were the only categories whose responses revealed mental mechanisms considering range vectors as objects of sets (thus, acting at Object Stages). This notion was conducted accurately in category 3, but not so in category 4. Even though, in category 4, mental mechanisms were considering vectors as objects of a range set, acting at Levels closer to Object Stages, they were faulty mechanisms incorrectly recognizing range vectors as those of the domain vectors not in the null space (or not similar to the structure of vectors of the null space). In fact, participants, in this category, appeared to have skipped entirely the ideas included in categories 1-3, specifically, the consideration of representations of matrix multiplications.

We also found that approaches employing the rref process (a very common approach in many matrix algebra textbooks in the US) may not be very efficient in helping learners form an accurate understanding of the range concept. In fact, responses (total 6) in category 5 are testimonials to the inefficiency of these approaches.

In summary, our findings point to mental mechanisms reflected in categories 1-3 as being the more effective ones toward the accurate formation and advancement of the knowledge of range ideas. Our findings, furthermore, point to desired (in the context of accurate knowledge structures displayed in participant responses) transitions between APOS Stages to be occurring with the consideration of representations of matrix multiplications, especially, in the case of linear combination representations of matrix multiplications. Even though the notion of evolving representations is not specifically addressed in this paper (a topic of future studies), we conjecture that changing characteristics of the representation of matrix multiplications may be a sign of evolving mental mechanisms acting on the representations.

Funding Agency

This work was supported by the University of Texas System and the Consejo Nacional de Ciencia y Tecnologia de Mexico (CONACYT) [Grant number 2019-35].

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