Exploring Geometric Reasoning with Function Composition

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Students learn about function composition, $(g \circ f)(x)$, in secondary school. From two given equations, one might identify the composite function algebraically via substitution, $g(f(x))$. But what about when functions are given as graphs? This study aims to explore how students were able to reason geometrically about composing functions. We identify common types of resultant graphs participants generated in trying to sketch the composition of various graphically-depicted functions, and the common types of reasoning we infer those participants engaged in while doing so. The results generate both questions, and hypotheses, about how best to support students' development of deep – and diverse – meanings for composing functions.

Keywords: function composition; graphs; geometric reasoning

Students are exposed to function composition in secondary school (cf., CCSSM, 2010). Primarily, they are taught that $(g \circ f)(x) = g(f(x))$ – meaning, for two functions defined by equations, one can procedurally determine the composite function via algebraic substitution. Additionally, in the context of inverse functions, secondary school students are taught that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$. Yet, such algebraic approaches are limited in their ability to develop sufficiently deep mathematical understandings of function composition. In this study, we try to explore the ways students' reason geometrically about the composition of two functions.

Background and Framework

In this study, we seek to answer the following research question: *In what ways do students graphically respond to function composition tasks?*

Function and Function Composition in Extant Literature

Much literature in mathematics education has been devoted to the function concept. This has included characterizing its essential abstract features (e.g., Freudenthal, 1983) and debating them (Mirin et al., 2021), considering its place in the school curriculum (e.g., McCallum, 2019; Paoletti & Moore, 2018), the notation we use for it (e.g., Paoletti et al., 2018; Thompson & Milner, 2019; Zazkis & Kontorovich, 2016), and studying how students and teachers understand and conceptualize it – including inverse functions (e.g., Breidenbach et al., 1992; Even, 1990; 1992). Scholars have also explored students' conceptions about real-valued functions (from $\mathbb{R}$ to $\mathbb{R}$) through the notion of covariational reasoning (e.g., Paoletti & Moore, 2017). In this context, a deep understanding of function demands students understand how quantities co-vary together – meaning, how changes in one quantity correspond to changes in the other quantity. However, studies have shown that students have difficulties in understanding the relationship between co-varying quantities and their graphs (e.g., Carlson et al., 2002; Schoenfeld, 1985). It can be concluded that there is a rich literature base for functions – including inverse functions.

Yet when it comes to function composition, very little exists. Much of the literature about composition of functions is in relation to calculating derivatives via the chain rule (e.g., Clark et al., 1997). In terms of covariational reasoning, function composition requires students understand how the quantity represented by the variable x covaries with the doubly-transformed output

$g(f(x))$, but this has been a minimal focus in terms of research (e.g., Moore & Bowling, 2008). Perhaps the study most closely related to function composition, on its own, was Ayers et al.'s (1988) exploration of using computer programming to help students develop the notion of function composition. In sum, there is a dearth of literature on function composition (especially in comparison to that about the function concept) – and none of it has considered it through a geometric lens. The study in this report aims to address this gap in the literature.

Geometric and Algebraic reasoning

A key premise underlying this research is that algebraic and geometric reasonings are different, where each provide distinct conceptions that are complementary for developing deep mathematical meanings. Broadly speaking, algebra is a mathematical field that examines particular structures, based on a set of objects (e.g., $\mathbb{R}$), and a binary operation(s) defined on them (e.g., $+$). Whereas geometry explores spaces that are related with distance, shape, and size (e.g., the Euclidean plane). To highlight this difference, a function, algebraically, is often characterized by its equation, e.g., $f(x) = 2x + 1$ (i.e., the structure of the set of points included in the relation); whereas, geometrically, we depict a function as its graph (typically in the Euclidean plane), e.g., $f(x)$ is a line. Moreover, not only do these fields explore different things, but key ways of reasoning differ between them. Driscoll et al. (1999; 2007), for example, differentiated Algebraic from Geometric "habits of mind." In our context, for simplistic purposes, we use *geometric reasoning* to mean reasoning that relies on figures and shapes in the plane and their properties; *algebraic reasoning*, by contrast, would rely on equations.

Beyond just the ways we might think about functions from school algebra, abstract algebra provides yet another point of connection to function composition. A primary connection for prospective teachers between abstract algebra and school mathematics is recognizing how functions and inverse functions are situated within their group structure (Wasserman, 2016). Notably, this structure is related through an operation on functions – namely, function composition. That is to say, in order for teachers to genuinely understand this mathematical connection between their university studies and the mathematics they will teach, they need to view composition as an "operation" in the same way that they view, for example, multiplication as an operation. They need intuitions, tools, ways of reasoning, and so forth, in order to be able to consider what the "result" of that operation of composition would be for two particular functions – even if those functions are given as graphs. Dreyfus and Eisenberg (1983) pointed out that without a formula, the graphical representation of function has very little meaning for most students entering calculus. This can be challenging, even for the mathematically adept.

Methodology and Data Sources

As a way to explore how students reason geometrically about function composition, we had (n = 143) university students in two mathematics and mathematics education programs (primarily from precalculus classes) sketch what they thought the composite function $(g \circ f)(x)$ would look like, primarily based on two given graphs of functions.

Task Design

On Desmos, participants were asked to sketch, using digital tools and within a limited timeframe (30 seconds, to capture their intuitions and reasonings), the composite function $(g \circ f)(x)$ for six pairs of functions. The majority of these pairs were represented only graphically (Figure 1 displays an example Desmos page; Figure 2 displays the other 5 tasks). The purpose in providing participants with somewhat unusual, and non-algebraic forms of the functions, was to deter them from immediately converting to an algebraic equation to complete the composition. Doing so increased the likelihood that students would have to try to reason about function composition geometrically based on the shapes of the graphs. In the last two tasks, functions were given as an equation, and as a table, as a point of comparison. In the complete study, participants: i) give an initial sketch within 30 seconds; ii) provide a written explanation for how, and why, they sketched the composite function the way they did; and iii) had a final opportunity to go back and modify any of their original sketches. In this paper, we analyze just their initial sketches. We do not include the "correct" composition $g(f(x))$ for the tasks in this paper, in part because it was uncommon for participants to sketch the correct graph but also to point out the genuine challenges of reasoning geometrically about function composition to the reader.

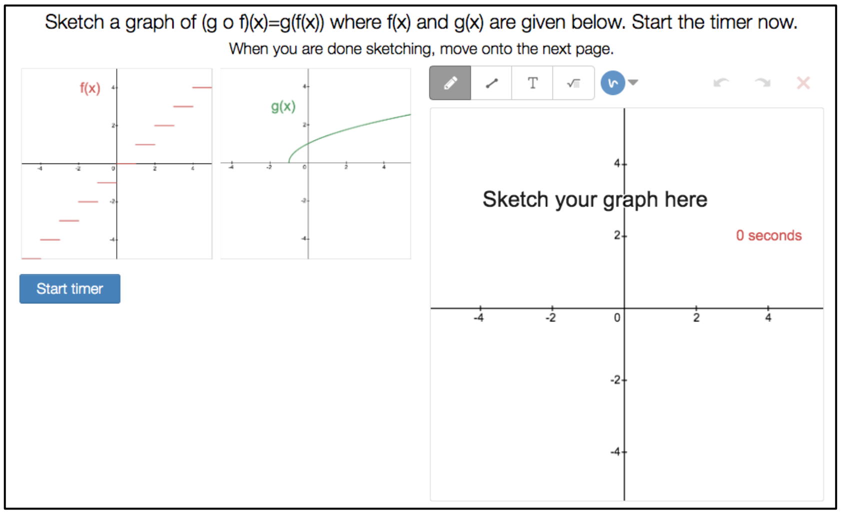

Figure 1. Task 2 in Desmos

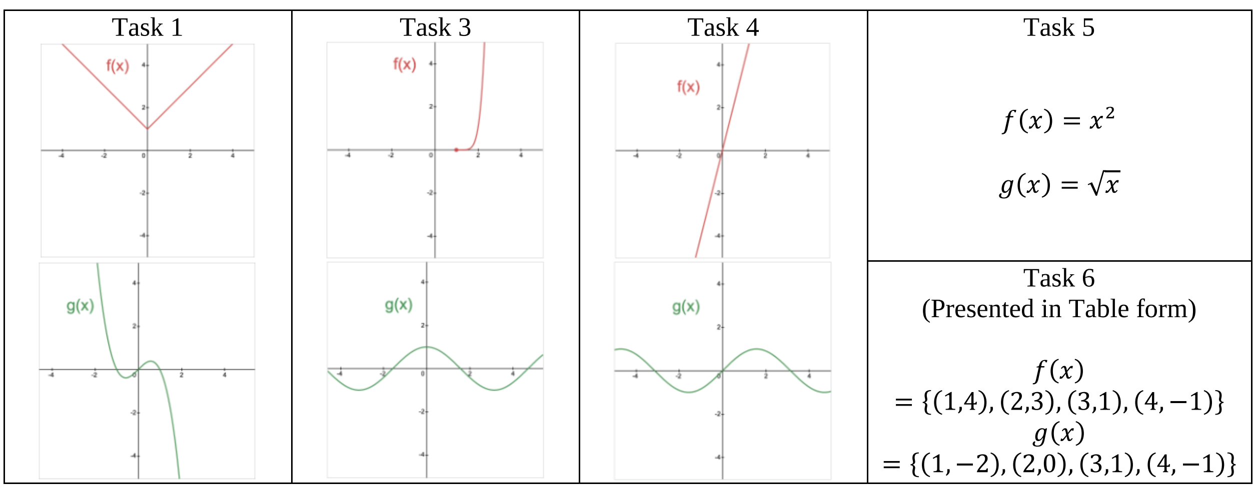

Figure 2. Given functions for Tasks 1, 3, 4, 5, 6

Analysis and Coding

Our analysis identified normative types of graphs, and processes for sketching those graphs, that were shared across participants. Notably, we aimed to create categories that could be applied across the entire set of tasks, although a few categories were specific to particular tasks. We anticipated these to include both incorrect and correct resultant graphs of composite functions, and a diverse range of reasoning students may have used to approach this sketching. We generated and refined all categories of coding as a group; when there were uncertainties in how to code a graph, we resolved such cases together and made refinements to categories as needed. As a rule of thumb, given that these were sketches, using a digital tool, we were reasonably lenient in trying to give students appropriate credit for their resultant graphs.

Due to space constraints, we do not include all codes used in our analysis. However, Table 1 provides codes and definitions for the majority of codes – and, in particular, those we report on in this paper. Clarifying examples are given in the results section.

Table 1. Codes for analyzing data

Code	Description of the composite function $g(f(x))$ sketched
BLANK	No graph
DOUBLE	Both graphs on top of each other
ONE	One of the graphs
ONE-T	One of the graphs, but with some basic transformation
MIXED HALF	Half of one graph put together with half of other graph
MIXED PROP	One graph put together with property of other graph
INVERTED	Correct graph of $f(g(x))$
CORRECT	Correct graph of $g(f(x))$
Y=X	Graphed the line $y = x$
OTHER	Some other graph

Results

Table 2 provides an overarching summary of students' initial ways of reasoning about sketching the composite function on the six tasks. Notably, we only report on the codes given in Table 1, so most, but not all (n = 143), of students' responses are present. Then, we look more carefully at results from the three types of tasks: (i) given graphs (Tasks 1-4), (ii) given equations (Task 5), and (iii) given tables (Task 6).

Table 2. Summary of data on Tasks 1-6

Code	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
BLANK	20	24	26	25	24	33
DOUBLE	26	22	31	34	19	26
ONE	10	12	10	15	10	26
ONE-T	31	16	17	19	5	0
MIXED HALF	2	6	12	8	4	1
MIXED PROP	9	26	12	12	0	0
INVERTED	1	0	0	2	3	0
CORRECT	7	5	1	4	27	3
Y=X	0	0	0	0	25	0
OTHER	27	27	30	20	24	48

Given Graphs

In Tasks 1-4, students were given *graphs* of $f(x)$ and $g(x)$ and asked to sketch the composite function. Although there are some differences between the four tasks, for the most part, we observed consistent themes in the different ways students approached these tasks. As a primary, overall result, students had very limited success in reasoning geometrically to produce a correct composite function. Only 3% of the responses (17 of 572) were correct, whereas roughly 35% were left “blank” or given some “other” unusual response. (All percentages given in this section are across all responses to Tasks 1-4.)

More interesting are the results for the different ways students appeared to reason geometrically about the result of composing two given graphs. Although we saw both DOUBLE and ONE codes on the tasks with given equations and tables, these were prominent ways of reasoning geometrically as well; and the transformation of one (ONE-T), and the mixing of both graphs (MIXED HALF, and MIXED PROP) were almost entirely observed in the graphical context.

The DOUBLE category was reasonably common (20% of responses) – meaning, participants frequently understood how function composition “combines” two graphs to be simply “put both functions on the same graph.” Participants also opted to just graph ONE of the graphs in about 10% of responses. Figure 3 gives a characteristic image for both of these from Task 2.

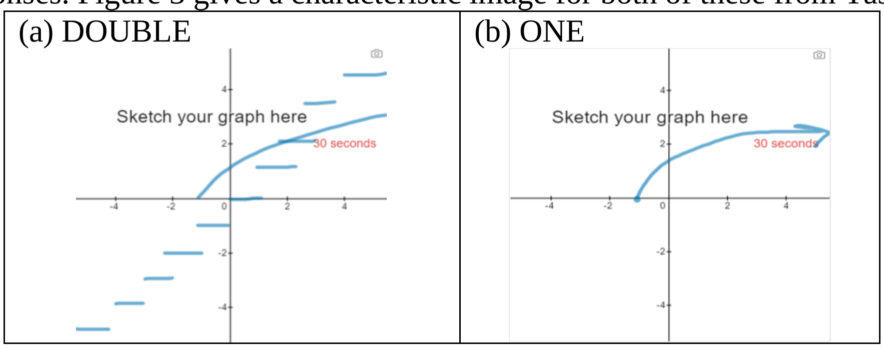

Figure 3. Examples from Task 2 of (a) “double” reasoning, and (b) “one” reasoning

Other common responses – and, in this case, ones there were mostly particular to the functions being given as graphs (Tasks 1-4) – had to do with modifying one or both functions in some way based on graphical features. This sort of geometric reasoning was how students interpreted “combining” two functions (via composition) given as graphs. For the ONE-T code (15% of responses), some students took one function as the base and then applied some transformation to it. Figure 4a shows one example, from Task 2, where a student took the initial $g(x)$ function (approximately square root) and then translated it by a vector of approximately $\langle 1,1 \rangle$, likely inspired by the step function going over one and up one. Although this is one example, there was quite a variety of different sorts of transformations included in this category – sometimes reflections, rotations, etc. For the MIXED HALF code (5% of responses), rather than graphing both functions at once, some students took half of one and combined it with half of the other to create a function – generally, a “left” and a “right” half. Figure 4b shows an example (also from Task 2). The MIXED PROP code (10% of responses) was one in which students applied a property of one function to the other function. As an example (from Task 2), Figure 4c shows one student who took the “disconnected dash” property of the step function $f(x)$, and applied it to the $g(x)$ function (approximately square root), to create a dashed square root graph. Again, there was a lot of diversity in the property being applied within this category – e.g., the domain of one was applied to the other, the pointiness of an absolute value function applied to the curves of the other, and so forth.

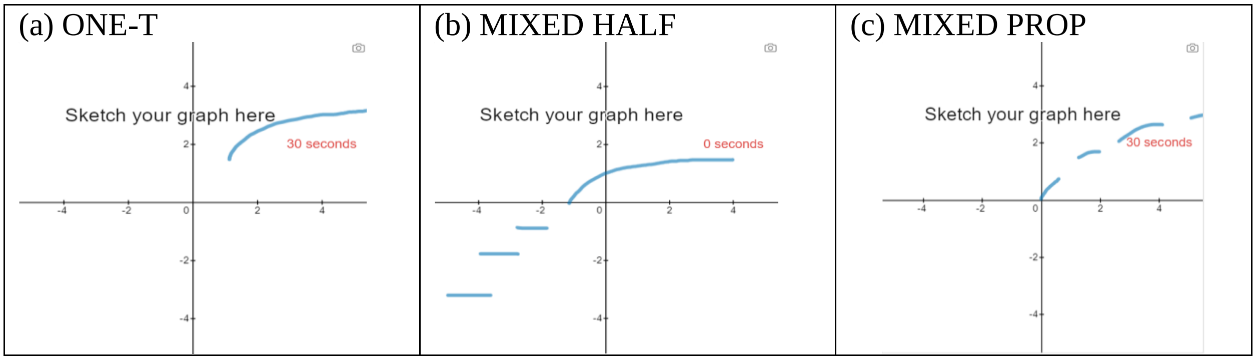

Figure 4. Examples from Task 2 of (a) "one-t" reasoning, (b) "mixed half" reasoning, and (c) "mixed property" reasoning

Given Equations

For Task 5, with two given *equations*, we point out two major results. Participants had the most success correctly plotting the composition of two functions given their equations (19%), which was significantly more than on any other task. One of the unique aspects of Task 5 was that the two functions would also have been known to be inverse functions. This may have simplified the task somewhat, as the output should be the function $i(x) = x$. But a particular nuance of this task is that the correct answer for the composite function is $g\big(f(x)\big) = |x|$ – it is $y = x$ only on a restricted domain ($x \geq 0$), and then with the reflective symmetry of the squaring function applied to it. Hence, these participants likely did more than just algebraic substitution to come up with their sketch. Another 18% of responses just graphed the line $y = x$, likely drawing on just their algebraic sensibilities. (This Y=X code was unique to this Task, as inverse functions were not given in any of the other tasks.)

Given Tables

We initially hypothesized that giving two functions in *table* form (Task 6) might facilitate students' ability to compute the composition of two functions; starting with one value, and then doubly-transforming it through f and then g can be readily accomplished with tables of values. This did not bear out in the data. Students struggled to find the composition, with about 33% of responses being some "other" unusual graph–almost twice as many of this category as on each of the other five tasks.

Conclusions and Implications

In this section, we summarize some of the important conclusions and discussion points based on the data. We highlight three important takeaways for the reader.

First, as an answer to the research question about the ways in which students reason geometrically about function composition, the data suggest students had significant difficulties sketching the correct composite function. We reiterate, on tasks where participants were given graphs (and tables), almost no students were able to give correct sketches. More were successful given equations, but even still, roughly the same number of participants gave correct responses as left it blank. This finding suggests that students are not being provided sufficient opportunities to learn about sketching composite functions overall, and especially from graphs and tables. Moreover, in the results, we described the five primary ways students tried to reason geometrically about what "combining" two functions (via composition) would look. These were: both graphs (DOUBLE), one graph (ONE), transforming one graph (ONE-T), half of one and half of the other (MIXED HALF), and applying a property of one to the other (MIXED PROP). Combined, these forms of reasoning accounted for approximately 70% of non-blank responses (58% of total responses).

Second, and perhaps unsurprisingly, students seem to be most familiar with composing functions in the context of two given equations – in particular, familiar functions, and whose composition they knew how to graph. This is the task in which students had the most success. Even still, we saw some of the challenges and limitations of algebraic reasoning in terms of producing a completely correct composite function – i.e., simplifying $\sqrt{x^2}$ to be x. This suggests the importance of having other ways to reason about function composition, specifically including geometric ways of reasoning from the graphs and their properties, which might be a useful way to avoid such oversimplifications. Although the input-output nature of functions may be on full display with tables, it was interesting to observe that students struggled with composing functions when given two tables. This generated some possible hypotheses as to why students struggled to make sense of the composite function being given tables: i) students experiences learning about composition were so strongly grounded in algebraic equations that they do not maintain ways of reasoning for composition in other contexts; ii) the discrete nature of the tables posed difficulties – because the collections of points, and their composition, would be disconnected (or possibly even undefined) on a graph; and iii) the quantified nature of the variable x in functions, where, unlike substituting the expression $f(x)$ into $g(x)$ *once*, the table form requires that students complete *multiple* substitutions – for *each* a in the domain of f, *each* of the $f(a)$ values needs to be substituted into $g(x)$ to obtain the composite collection of points.

Third, as a final point, we elaborate on why we think these particular findings are important. One reason is that the study data collected here document the genuine challenges students face in sketching the composition of two functions – in almost any form they are given, but especially graphically and tabularly. This finding provides a basis upon which to build future work. It suggests students have not been provided opportunities to develop the intuitions, ways of reasoning, and tools needed to develop a robust concept of function composition; in particular, a way to understand function composition as an operation – something that, in some way, "combines" two given functions. A second reason these findings are important is because they document the various ways that students might try to reason geometrically about function composition. Although most of these are non-normative, and would not result in a graph of the actual composite function, they provide a launching point on which instructors can build more normative conceptions. Notably, composite functions often do resemble their initial parent functions – taking, in some form, some aspect of the shape or properties of those initial functions. Third, the findings here point to both the challenges, as well as the unexplored opportunities, to identify ways to advance and enrich students' conceptions of function composition. More work is needed to identify productive intuitions and way of reasoning that might be helpful to build on, and to explore representations and digital tools that might afford learners an opportunity to develop a rich conception of function composition. In the same way that students develop meaningful ways to think about arithmetic operations like addition and multiplication, developing ways to recognize function composition as an "operation" in a similar manner is an important aim in mathematics education – in large part because of the universality of the function concept, and the utility of this operation of composing them.

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