The Role of Language in Undergraduate Mathematics: Linear Independence and Limit

Ernesto Daniel Calleros
San Diego State University and University of California San Diego

As English is increasingly seen as the language of mathematics instruction in the United States and the world, it becomes important to interrogate the role of the English language in mathematical understanding. This article investigates this question in relation to linear (in)dependence and limit through clinical interviews with undergraduate students. Preliminary results indicated a relationship between students' conceptual understanding of linear (in)dependence and their ability to connect their mathematical and non-mathematical uses of the term (in)dependent. Moreover, students' unproductive conceptions of limit appeared to be influenced by non-mathematical meanings of limit.

Keywords: Linear Algebra, Calculus, Language, Student Thinking, Undergraduate Education

As English is increasingly seen as the language of mathematics instruction in the United States and the world (e.g., Gerber et al., 2005; Planas, 2018), it is crucial to interrogate the role of the English language in mathematical understanding. Prior research suggests that the structure of the English language may be less effective than some other languages, say the Chinese language, for facilitating mathematical performance at primary and secondary school levels (e.g., Han & Ginsburg, 2001). This paper investigates the role of the English language at the undergraduate level in the context of limits and linear independence, as these are two central topics of study in mathematics education (e.g., Larsen et al., 2017; Rasmussen & Wawro, 2017).

Literature Review

Most English mathematics words at the K-12 levels have been created from Greek or Latin (Milligan & Milligan, 1983). For example, the term *quadrilateral* comes from two Latin words: *quadri* which means "four" and *latus* which means "side." Unfortunately, most English-speaking students are not taught Latin or Greek, so many of them may come to see math terms as random combinations of letters with no inherent meaning (Johnston-Wilder et al., 2016). In contrast, Chinese math words appear to have been created as descriptive and conceptually clear compound phrases (Han & Ginsburg, 2001). For example, the Chinese term for *quadrilateral* literally translates into English as *four sides shape*, which clearly depicts the concept of quadrilateral.

Yet, many undergraduate math terms in English such as *linearly independent* already seem conceptually clear to the expert's eye. But are such terms also conceptually clear to the undergraduate learner? Research on linear (in)dependence and limit documents a diversity of student conceptions of these topics. For example, Plaxco and Wawro (2015) analyzed data from clinical interviews with students from a linear algebra course that used a travel-oriented curriculum (Wawro et al., 2012). They identified four student conception categories of linear (in)dependence: travel, geometric, vector algebraic, and matrix algebraic. The travel conception described linear (in)dependence in terms of movement. The geometric conception used language about spatial reasoning or graphical representations. The last two conceptions focused on operations on algebraic representations of vectors and matrices, respectively.

Williams (1991) investigated the following student models of limits: (a) dynamic-practical, (b) acting as a boundary, (c) formal, (d) unreachable, (e) acting as an approximation, and (f) dynamic-practical. He found that it was common for students to exhibit a dynamic view of limit and organized this view into three variants: squeezing a value from both sides, finding approximations, and coordinating changes in x with changes in y. Building on William's (1991) work, Oehrtman (2009) characterized students' spontaneous metaphors for limits and identified five often used student metaphors of limit: as physical limitation, with infinity as number, as proximity, as collapse, and as approximation. He also identified three less often used student metaphors of limit: as motion, as zooming, and as informal version of correct definition. Of all these metaphors, Oehrtman found limit as an approximation to be the most productive.

My study investigates the uses of the terms *limit* and *(in)dependent* in relation to student understanding of limits and linear (in)dependence. Here, student understanding is conceptualized as any or all of the following elements of mathematical knowledge: meanings, images, ideas, connections, ways of comprehending situations, and explanations (Lobato, 2014). With this framework in mind, the research question is: What is the role of language in student understanding of linear (in)dependence and limits?

Methods

To gather background information on student understanding of linear (in)dependence and limits, the research team conducted clinical interviews (Ginsburg, 1997) with two groups of undergraduate students across two universities in the southwest of the United States. One group consisted of 4 students who had recently taken linear algebra and was asked questions about their understanding of linear (in)dependence. The second group had 6 students who were enrolled in Calculus II or III and were asked about their understanding of limits. Student interview responses were transcribed and analyzed. Students' linear (in)dependence conceptions were inferred from the data using open coding from grounded theory (Strauss, 1987). To infer students' conceptions of limits, we used a list of a priori codes from prior research (e.g., Williams, 1991), while allowing room for other codes to emerge from the data within a grounded theory approach (Strauss & Corbin, 1990).

Next, we analyzed the previously identified student understandings of limits and linear (in)dependence, considering the role of language. First, we determined which, if any, student model of limit related to the non-mathematical meaning of *limit.* Then, we analyzed student responses to the interview question regarding why the words *independent* and *dependent* are used for linear (in)dependence. In each response, we determined if the student used meanings of (in)dependence from outside linear algebra to interpret linear (in)dependence. Finally, we analyzed the relationship between each student understanding of linear (in)dependence and their use of outside meanings of (in)dependence.

Results

Background Findings: Student Understanding of Linear (In)dependence and Limits

Three student models of linear (in)dependence emerged: (a) A set of vectors (viewed as a matrix) is linearly independent if and only if it is reducible to the identity matrix; (b) A set of vectors is linearly independent if and only if no two vectors in the set are scalar multiples of each other; and (c) A set of vectors is linearly independent if and only if at least one vector is a scalar multiple of another vector in the set. Student models (a) and (c) were used by one student each (Marco and Arletta, respectively), whereas student model (b) was used by two students (Edward and Alana). While none of these student models was completely correct mathematically, some were more correct and conceptually oriented than others. Student model (a) was more computationally oriented, focused on reducing a matrix through row or column operations. On the other hand, models (b) and (c) highlighted connections between linear (in)dependence and the idea of scalar multiples, with model (b) being more logically correct than (c).

Regarding the limit analysis, each student had multiple conceptions of limit in calculus. We identified eight student models of a function's limit: Coordinating x and y (n=6 students), approaching from both sides (n=5), function value (n=5), continuity (n=3), boundary (n=4), unreachable (n=6), exception (n=2), and stopping point (n=1). The model of limit *coordinating x and y* described a limit in terms of how the function value changes as x approaches a certain value. *Approaching from both sides* described 'limit as x approaches *a*' as the common y-value the graph approaches from the left and from the right of *a*. *Function value* treated the limit value and the function value as the same. *Continuity* viewed the limit's existence as equivalent to the function's continuity. *Boundary* conceptualized a limit as a number or point past which the function cannot go. *Unreachable* conceived of a limit as a number or point that cannot be reached by the function graph. *Exception* viewed the limit as something that does not follow a rule. Finally, *stopping point* conceptualized the function as motion along a graph and viewed the limit as the place where the function is constrained.

The Role of English on Student Understanding of Linear (In)dependence

The results indicated that Edward and Alana, the two students with model (b) of linear (in)dependence, connected their non-mathematical understandings of the terms *independent* and *dependent* to their understanding of linear (in)dependence, whereas the other students (Arletta and Marco) failed to make such a connection.

For example, when I asked Edward, "We've been talking about linearly independent sets of vectors…Why do you think the words 'independent' and 'dependent' are used?," he responded, "Well, dependent means they're *related* to each other. Right? Independent means they're not." Then, when I asked him to elaborate, he said, "Well, dependent means it's related to or reliant upon each other. So, if you have a dependent set of vectors, you can make one vector from another by scaling it. So, they're interconnected, I guess." Based on this excerpt, it appears Edward connected his understanding of the term *dependent* as used outside linear algebra (i.e., "related to or reliant") to his interpretation of that term in linear algebra (i.e., "make a vector from another by scaling it").

Similarly, when Alana was asked the same question, she said:

> Okay, I like that the words are the words they are, because they help me remember the definition. So, when it's dependent– just like, in general the definition of dependence, to me, is when you need something else, or, like, you rely on someone else. So, if you just, like, rely on something or someone– so then, thinking about it mathematically, like– I'll just use this example I've been using. Oops. This [vector] times two gives you this [vector], and so there's that, like, *relationship*. And then independence, to me, in general means completely on your own. So, like, there's no way that these two [vectors] can ever, like, work together and be the same or look the same.

Thus, she used her general knowledge from outside of mathematics that "dependence…is…when you need… or rely on someone else" and that independence means "completely on your own" to help create mathematical interpretations or examples of linear (in)dependence (e.g., "this [vector] times two give you this [vectors]" for dependence and "no way these two [vectors] can ever look the same" for independence).

In contrast, Marco did not leverage his understanding of the words *independent* and *dependent* from outside of mathematics for interpreting linear (in)dependence. When asked the same question above, he responded:

> Yeah, honestly, I've been going through the whole… linear algebra course. I didn't know why it was called independent and dependent. I just thought, well, if it reduces to identity matrix, it's independent. That's all the correlation I got from that.

Hence, Marco claimed that, in interacting with the concept of linear (in)dependence, he had not attended to the meanings of *independent* and *dependent* outside linear algebra. Not connecting his non-mathematical and mathematical understandings of *independent* and *dependent* might have led Marco away from attending to scalar multiples (for interpreting linear (in)dependence) and instead into rote learning.

Like Marco, Arletta initially claimed never connecting her prior linguistic understanding of (in)dependence to linear (in)dependence. However, when I later presented her with a traditional textbook definition of linear (in)dependence, she forcedly made a connection: "The words independent and dependent [are used] because they're depending on whether or not the input values are zero to get a[n] output of zero." Moreover, when I pressed her to exemplify this connection, Arletta responded:

> Um, I guess, just going back to the definition. Like if c_1x_1— Like if c_1, c_2 [the coefficients] were to equal 0, then that means they're [the vectors x_1, x_2] independent… and... in dependent ones [vectors], these values [the coefficients] can be anything, but they don't all have to be zeros.

Although Arletta made an interesting connection between the formal definition of linear (in)dependence and her prior understandings of terms *(in)dependent*, this connection was superficially (not conceptually) aligned with the topic of linear (in)dependence. This is not surprising, given that she admitted not connecting the formal definition to her understanding of linear (in)dependence. This may help explain why, in her model of linear (in)dependence, she labeled a set with at least two vectors which are scalar multiples of each other as linearly independent and a set with no scalar multiples as linearly dependent (which is mathematically reversed).

These results indicate that the students with the strongest conceptual understanding of linear (in)dependence (those with model (b)) were precisely those who connected their mathematical and non-mathematical understandings of the words (*in)dependent* in terms of a *relationship* between vectors.

The Role of English on Student Understanding of Limit

All six students appeared to connect their mathematical and non-mathematical conceptions of limit. All students had at least one mathematical conception of limit that related to the non-mathematical definition of limit as a restraint or limitation – "a measure or condition that keeps something under control or within limits" (2019, n.p.). This non-mathematical definition aligns with the following (mathematically incomplete or incorrect) student models of limit inferred from the data: boundary (n=4), unreachable (n=6), exception (n=2), and stopping point (n=1).

To illustrate the connection between the non-mathematical meaning of limit and student mathematical understanding of limit, consider one student, Joyce, who had a *boundary* model of limit. When Joyce was asked what came to mind when she heard the word *limit*, she said, "When something cannot be something anymore like 'I'm at my limit.'" Then, to elaborate, she sketched a graph and its highest point while saying, "Here's a dot… and there's like a function… and it's like a maximum, but I guess like, that's as high as it goes. Like that's what I'm picturing like it's the limit… like the dot [the highest point on the graph] is the limit." Thus, Joyce related her non-mathematical conception of *limit* as a limitation (e.g., "like I'm at my limit") to her mathematical understanding of limit as a function's "maximum" value (which is mathematically incorrect).

Discussion

Overall, the results indicated that students' non-mathematical language can play an important role in students' mathematical understanding of linear (in)dependence and limits. In particular, it may be that connecting the non-mathematical use of (in)dependence as a *relationship* to the mathematical meaning of linear independence afforded students a deeper understanding of this topic. However, not all students in the study who had taken linear algebra made such a connection, suggesting that there is still room for making the English term *linearly (in)dependent* conceptually clearer to students. On the other hand, the analysis of student understanding of limits suggested that extending non-mathematical understandings of the term *limit* to mathematical interpretations of this term without reflection may also direct students' mathematical understandings astray.

These results highlight the importance for teachers to help students reflect on the terminology used to describe the concepts they are studying. This reflection may encourage students to compare the mathematical and non-mathematical meanings of certain terms. Leading students to see conceptual similarities (when they exist) can allow students to build on their own linguistic resources, whereas encouraging them to see differences can re-direct students away from unproductive mathematical conceptions rooted in limitations of non-mathematical meanings.

For example, a linear algebra instructor could facilitate discussions about the non-mathematical meaning of *dependent* and its connections, not only to the idea of scalar multiples of vectors, but more generally to the concept of linear combinations. This could help students interpret the idea of "a vector being expressible as a linear combination of other vectors" as a relationship in which that vector "depends" or "relies" on the other vectors. Likewise, a calculus instructor could facilitate discussions about the similarities and differences between mathematical and non-mathematical meanings of the term *limit*, so students may confront potentially unproductive conceptions and build productive models of *limit*.

To prompt or enrich such reflections, features of other languages, such as Chinese, could be leveraged. For example, if linear algebra students don't bring up the idea of a "relationship" on their own in class discussion, the instructor could promote such idea by asking students to consider the literal translation of the Chinese term for *linearly dependent* (线性相关: linearly related to each other*)*, which explicitly invokes the idea of a relationship. Similarly, to enhance the discussion about limits in a calculus class, students might be asked to reflect on the constrains of the literal translation of the Chinese name for *limit* (极限: maximum magnitude limit). This literal translation appears to reinforce the mathematically incorrect idea (held by students like Joyce above) that a limit is the maximum value of a function.

Ultimately, whether the multiple meanings embedded in certain mathematical terms support or hinder student mathematical understanding may lie on how students and instructors leverage these meanings. One promising way is for instructors to promote student reflection over the similarities and differences between multiple meanings of important mathematical terms. More research in language and undergraduate mathematics may shed more light on this promise.

Acknowledgments

I would like to thank Dr. Amelia Stone-Johnstone and Mike Foster, my colleagues and friends, for contributing to the design and implementation of the clinical interviews on limits and the initial coding of the data on limits.

References

Gerber, A., Engelbrecht, J., Harding, A., & Rogan, J. (2005). The influence of second language teaching on undergraduate mathematics performance. *Mathematics Education Research Journal*, *17*(3), 3–21.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Han, Y., & Ginsburg, H. P. (2001). Chinese and English mathematics language: The relation between linguistic clarity and mathematics performance. *Mathematical Thinking and Learning, 3*(2-3), 201–220.

Johnston-Wilder, S., Lee, C., & Pimm, D. (2016). *Learning to teach mathematics in the secondary school: A companion to school experience* (4th ed.). London: Routledge.

Larsen, S., Marrongelle, K., Bressoud, D., Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J Cai (Ed.). *Compendium for research in mathematics education* (pp. 526-550). Reston, VA: National Council of Teachers of Mathematics.

Lobato, J. (2014). Why do we need to create a set of conceptual learning goals for algebra when we are drowning in standards? In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emergent models of students' algebraic knowing*, WISDOMe Monographs (Vol. 4, pp. 25-47). Laramie, WY: University of Wyoming.

Limitation. (2019). In *Merriam-Webster.com*. Retrieved May 16, 2019, from https://www.merriam-webster.com/dictionary/limitation

Milligan, C. F., & Milligan, J. L. (1983). Sharing teaching ideas: A linguistic approach to learning mathematics vocabulary. *The Mathematics Teacher, 76*(7), 488–490.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.

Planas, N. (2018). Language as resource: A key notion for understanding the complexity of mathematics learning. *Educational Studies in Mathematics, 98,* 215-229.

Plaxco, D., & Wawro, M. (2015). Analyzing student understanding in linear algebra through mathematical activity. *Journal of Mathematical Behavior, 38,* 87–100.

Rasmussen, C., & Wawro, M. (2017). Post-calculus research in undergraduate mathematics education. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 551-581). Reston, VA: National Council of Teachers of Mathematics.

Strauss, A. L. (1987). Qualitative analysis for social scientists. Cambridge: Cambridge University Press.

Strauss, A. & Corbin, J. (1990). *Basics of qualitative research.* Newbury Park, CA: Sage Publications.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*(3), 219-236.