Synthetic vs. Analytic Formalism of Matrix Representations in Linear Algebra: What it Means to Diagonalize a Linear Transformation

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The contributed report highlights linear algebra students' understandings of the matrix representations of certain linear transformations defined on with respect to given bases in a DGS-MATLAB assisted pedagogical environment. The research focused on the diversity of strategies that students utilized specifically in the process of diagonalizing linear transformations in a series of in-depth qualitative interviews within a framework of modes of description and thinking in linear algebra (Dreyfus, Hillel & Sierpinska, 1998; Sierpinska, 2000). Data analysis revealed three main categories: (i) synthetic-geometric mode manifested as the visual alignment strategy in which students visualized each image vector as a multiple of the corresponding preimage vector; (ii) analytic arithmetic mode prevailed in the process of expressing the image vectors in terms of the preimage vectors; (ii) analytic structural mode was observed in the process of determining the diagonal form of the linear transformation with reference to the transition matrices and the commutative diagram.

Keywords: matrix representation of linear transformations; transition matrices; diagonalization; eigenbasis; dynamic geometry software; modes of description and thinking in linear algebra

Theoretical Background and Motivation for the Study

The pedagogy of linear algebra has been a focus of research interest for many years (Dogan-Dunlap, 2010; Dorier, 1991, 1995, 1998; Dorier, Sierpinska, 2001; Gueudet-Chartier, 2006; Harel, 1987, 1989, 1990; Parraguez & Oktac, 2010; Pavlopoulou, 1993; Robert & Robinet, 1989; Rogalski, 1994; Sierpinska, 1995, 2000; Sierpinska, Dreyfus & Hillel, 1999; Sinclair & Gol Tabaghi, 2010; Thomas & Stewart, 2011; Zandieh, Wawro & Rasmussen, 2017). Hillel (2000) characterized linear algebra students' understanding of matrices and linear operators to be at the interoperational level of thinking – although they were expected to communicate at the transoperational level of thinking. Hillel (2000) further posited the requirement that at the trans-level linear algebra learners should be able to: (i) determine the matrix representation of a linear transformation relative to a given basis; (ii) determine the coordinate representation of a vector in a vector space relative to a basis; (iii) think about these representations (coordinate representations of vectors and matrix representations of linear transformations) as "objects of inquiry in their own right;" (iv) think about "the general conditions under which a vector or a linear operator can have a particularly desirable representation" (p.206).

Harel and Kaput (1991), and Harel (1985, 1990) formulated the concreteness principle, as a fundamental approach for the teaching and learning of linear algebra, originated from Piaget's (1977) idea of conceptual entities. According to this principle, "for students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say the student has mental procedures that can take these objects as inputs" (Harel, 2000, p.180). Concreteness principle requires that "students build their understanding of a concept in a context that is concrete to them" (p.182). He recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle. Aligned with the concreteness principle (Harel, 2000) and in search for a desirable matrix representation for a linear transformation as brought forward in Hillel (2000), this study focuses on students' questioning of the essence of diagonalization of linear transformations in a DGS-MATLAB-assisted learning environment.

Modes of Description and Thinking in Linear Algebra

Representations are central to the teaching and learning of linear algebra. Harel (1989) established, among other things, that geometric representations contribute significantly in linear algebra students' concept image formation (p. 57). The present report is guided by the theoretical viewpoint that identifies three modes of description (language) with the corresponding three modes of thinking as demonstrated by linear algebra students (Dreyfus, Hillel & Sierpinska, 1998; Hillel, 1997, 2000; Sierpinska, 2000): (i) Geometric description (synthetic-geometric mode of thinking): point, line, plane, geometric transformation (translation, reflection, rotation, projection), vector as a directed line segment (arrow with a head and tail); (ii) Arithmetic description (analytic-arithmetic mode of thinking): n-tuples, matrices and operations on matrices including the inverse of a matrix, the adjoint of a matrix, partitioned matrices, systems of linear equations and their solutions; (iii) Algebraic description (analytic-structural mode of thinking): theory of vector spaces, subspaces, inner product spaces, linear transformations.

Context and Method

Qualitative-descriptive interview data were collected over three years in a university in the United States, as part of a research project which was designed for the purpose of enhancing mathematics and mathematics education majors' content knowledge of advanced mathematics, with particular focus on geometry-linear algebra connections using DGS-MATLAB technology. The data for the analysis of matrix representations of linear transformations concept came from the videotapes of interview sessions in a computer lab that included a total of sixteen math majors who successfully completed a one-semester long linear algebra course, who were interviewed by the author individually on separate days. The university linear algebra course math majors took had covered the first eight chapters of Ron Larson's "Elementary Linear Algebra" textbook. Math majors were familiar with the MATLAB and DGS as they had already used them during the preceding interview sessions on linear algebra topics; they were not provided any training or instructional help during the interviews.

Table 1. Interview tasks: Find the matrix A' for $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis β'.

Linear Transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$	Basis β'
$T(x,y,z) = (x+z, -x+3y+z, 2z)$	$\beta' = \{(2,1,0),(1,0,1),(0,1,0)\}$
$T(x,y,z) = (3x+2y+z, 2z, 2y)$	$\beta' = \{(1,-5,5),(-3,1,1),(1,0,0)\}$
$T(x,y,z) = (x+2y-2z, -2x+5y-2z, -6x+6y-3z)$	$\beta' = \{(1,1,3),(1,0,-1),(1,1,0)\}$
$T(x,y,z) = (2y, x-y, z)$	$\beta' = \{(-1,1,0),(2,1,0),(0,0,1)\}$
$T(x,y,z) = (3x+2y-3z, -3x-4y+9z, -x-2y+5z)$	$\beta' = \{(1,-3,-1),(3,0,1),(-2,1,0)\}$

Sample Interview Outline: Why did you decide to begin with the standard matrix A for T? How would you (What does it mean to) obtain the matrix A' for T relative to basis β'? Visually? Analytically? How are matrices A and A' related? What are your thoughts about the types of the matrices A and A' in relation to each other? What made you embrace the visual approach? The analytic approach? Is T diagonalizable? What does it mean for a linear transformation to be diagonalizable.

The qualitative interviews were based on a semi-structured interview model (Kvale, 2007) in the course of which the interviewer followed-up with probes and questions on the interviewees' responses. Math majors were asked to respond to a variety of interview questions with primary focus on the matrix representations of linear transformations in a manner leading to students' questioning of the essence of diagonalization. During the interviews, math majors were asked to think aloud, to clearly indicate their problem solving procedure, and to explain their reasoning in detail. Math majors were granted access to scratch paper along with MATLAB and DGS to facilitate and clarify their explanations during the interviews. All interview tasks were designed in such a way that they could be explored via both analytic and synthetic approaches, in accord with the guiding theoretical framework used in the study. Table 1 outlines the linear transformations along with an interview outline that were used during these interviews. Though it was optional, all math majors were very eager and passionate about using MATLAB and DGS, primarily for checking their work, visualizing analytic approaches, testing conjectures, or providing examples and counter-examples.

Analysis of data, which consists of videotaped qualitative interviews along with MATLAB-DGS work and inscriptions, was carried out using thematic analysis (Boyatzis, 1998). The thematic analysis was primarily used to describe how research participants came up with a diversity of innovative and creative ways for understanding and making sense of matrix representations of linear transformations and their connections to other important concepts of linear algebra. After transcribing all interview videos, using thematic analysis, the author reviewed the interview videos along with the original transcripts line by line to gain access to the modes of description and thinking adopted by students as they explored matrix representations of linear transformations and their relationship to other core concepts of linear algebra, in accordance with the research objective. The last cycle of data analysis process consisted of a holistic review of the corpus of data multiple times, in accordance with constant comparative methodology (Glaser & Strauss, 1967).

Results

Overall, students provided a diversity of synthetic and analytic strategies in their exploration of the diagonal form of linear transformations. This section considers the visual alignment approach (synthetic-geometric mode), the expressing of the image vectors in terms of the preimage vectors approach (analytic-arithmetic mode), and the transition matrix approach (analytic-structural mode), respectively. In each interview task, students were asked to find the matrix representation A' for the given linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to a specifically chosen nonstandard basis β'. Because most of the time all sixteen students began each task by retrieving the standard matrix A for T, the analysis that follows primarily focuses on what happened next.

Synthetic-Geometric Mode: The Visual Alignment Approach

Upon obtaining the standard matrix A for T, six out of sixteen students consistently embraced the synthetic-geometric approach in an organized attempt to arrive at the matrix A' for T relative to β'. Typical student approach was to first introduce A along with the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, respectively on the DGS. Second step was to obtain the *matrix-applied* basis vectors $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$, denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, respectively on the DGS (Figure 1). Upon noticing the parallel relationships $\mathbf{v}_1 \parallel \mathbf{u}_1, \mathbf{v}_2 \parallel \mathbf{u}_2, \mathbf{v}_3 \parallel \mathbf{u}_3$, visually, students in this category immediately focused on the scale factor $\lambda_1, \lambda_2, \lambda_3$, for each parallel vector pair, which, they mentally straightforwardly determined as $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, respectively. The final step was to simply write down the diagonal matrix $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

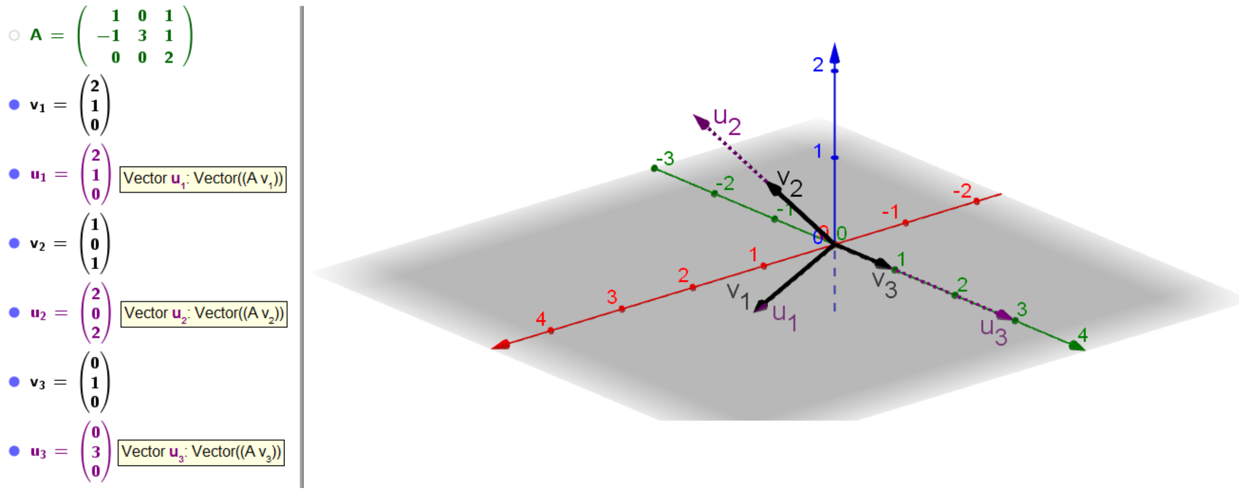

Figure 1 Typical student synthetic-geometric approach for Task 1.

Students' synthetic-geometric description of the parallel vector pair (i.e., the basis vector $\mathbf{v}_i$ and the matrix-applied basis vector $A\mathbf{v}_i$) relationship appeared to have functioned in two steps: (i) when $\mathbf{v}_i$ and $A\mathbf{v}_i$ were aligned in the same octant, the corresponding eigenvalue λ_i was given by the ratio of the length of $A\mathbf{v}_i$ to the length of $\mathbf{v}_i$; (ii) when $\mathbf{v}_i$ and $A\mathbf{v}_i$ were aligned in opposite octants, the corresponding eigenvalue λ_i was given by the negative ratio of the length of $A\mathbf{v}_i$ to the length of $\mathbf{v}_i$. Upon the interviewer's probing whether T is diagonalizable, S1 responded: "*Well in this case A' is a diagonal matrix so I would say yes T is diagonalizable*." S2, who embraced the visual approach like S1, was more specific in her explanations: "*Only because all these* [pointing to the image vectors] *are multiples of these* [pointing to the preimage vectors] *the linear transformation is diagonal*." In the synthetic-geometric approach, students simply focused on each image vector as a multiple of the corresponding preimage vector, without considering the remaining two preimage vectors. This was what made the linear transformation diagonal.

Analytic-Arithmetic Mode: Expressing the Image Vectors in Terms of Preimage Vectors
Six students consistently embraced the analytic-arithmetic mode. Among these six, three students directly applied the linear transformation to each vector of the given nonstandard basis, whereas the other three applied the standard matrix representation of the linear transformation to the coordinate vector representation of each basis vector. This was the slight difference in the thinking of these two groups of students.

Expressing $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. In Task 2, for instance, S3 first evaluated the *linear transformation-applied* basis vectors $T\mathbf{v}_1 = T(1,-5,5), T\mathbf{v}_2 = T(-3,1,1), T\mathbf{v}_3 = T(1,0,0)$ and obtained the corresponding image vectors $(-2,10,-10), (-6,2,2), (3,0,0)$. Simply by comparing the preimage-image vector pairs was she then able to retrieve the eigenvalues as

$\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = 3$, respectively via mere inspection. This was enough for her to deduce the desired matrix representation as $A' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Expressing $A[\mathbf{v}_1]_\beta, A[\mathbf{v}_2]_\beta, A[\mathbf{v}_3]_\beta$ in terms of $[\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, [\mathbf{v}_3]_\beta$. Among the three who embraced this *representation* approach, only one student felt the need to specify the coordinate vector representation notation $[\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, [\mathbf{v}_3]_\beta$ where β denoted the standard basis. The other two students, without being specific, still were able to interpret it as the representation of the vector *object*. S4, who embraced the representation approach for Task 3, explained: "*I'd first apply A to each basis vector*." Upon juxtaposing the three image vectors (written as 3×1 column matrices $\begin{bmatrix} -3 \\ -3 \\ -9 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$), respectively, she first suggested $A' = \begin{bmatrix} -3 & 3 & 3 \\ -3 & 0 & 3 \\ -9 & -3 & 0 \end{bmatrix}$ as the matrix representation of T relative to β'. Upon the interviewer's probing how she knew whether her suggestion was the desired matrix representation, she responded: "*I compare each column to the standard basis vectors… No, that's not right. I am supposed to be comparing them to the nonstandard basis vectors and it looks like each column is a factor.. I mean multiple of each basis vector so I am changing my answer to a diagonal matrix*." This way, S4 obtained $A' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ as the desired matrix representation.

Analytic-Structural Mode: The Transition Matrix Approach

Among the four consistently adopting the transition matrix approach, two students emphasized the matricial relationship by drawing a commutative diagram. Except for one student, S5, all students began each task by first writing the transition (change of basis) matrix as $P = \beta'$. S6, for instance, began her exploration of Task 4 by setting $P = \beta'$ while explaining: "*Okay I remember this very well.. this is the nice case.. P is the transition matrix from β' to β so I don't have to do anything I remember this very well*." With reference to her commutative diagram, she proposed the matricial equation $P^{-1}AP = A'$ and verified her proposed diagonal form $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on MATLAB (Figure 2a). Likewise, Figure 2b demonstrates S7's MATLAB procedure based on the transition matrix approach in her exploration of Task 2.

```
>> v1=[-1;1;0];v2=[2;1;0];v3=[0;0;1];
>> P=[v1 v2 v3];
>> A=[0 2 0;1 -1 0;0 0 1];
>> P^-1*A*P

ans =

   -2.0000   -0.0000         0
         0    1.0000         0
         0         0    1.0000
```

a

```
P =

     1    -3     1
    -5     1     0
     5     1     0

>> inv(P)

ans =

         0   -0.1000    0.1000
         0    0.5000    0.5000
    1.0000    1.6000    1.4000

>> A=[3 2 1; 0 0 2; 0 2 0]

A =

     3     2     1
     0     0     2
     0     2     0

>> inv(P)*A*P

ans =

   -2.0000         0         0
         0    2.0000         0
   -0.0000   -0.0000    3.0000
```

b

Figure 2 Transition matrix approach.

Difficulties Observed within the Three Modes of Description

This section highlights some of the difficulties students encountered in their exploration of the matrix representations of linear transformations in all three modes of description.

Retrieving the zero eigenvalue from the matrix applied eigenvector. This difficulty occurred in the synthetic-geometric approach in the exploration of Task 5. All six students who embraced the visual approach successfully graphed the given basis vectors along with the matrix-applied basis vectors. Obviously, $\mathbf{v}_2$ and $\mathbf{v}_3$ were both parallel to $A\mathbf{v}_2$ and $A\mathbf{v}_3$, respectively, in the same direction, with corresponding eigenvalues 2, and 2, respectively (Figure 3). It took quite some time for all students to realize that 0 was indeed the eigenvalue corresponding to $\mathbf{v}_1$. S8, for instance, felt the need to switch to the analytic-arithmetic mode to resolve his confusion regarding $\mathbf{v}_1$. In his first attempt, he used the matrix approach by writing $A\begin{bmatrix}1\\-3\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, A\begin{bmatrix}3\\0\\1\end{bmatrix}=\begin{bmatrix}6\\0\\2\end{bmatrix}, A\begin{bmatrix}-2\\1\\0\end{bmatrix}=\begin{bmatrix}-4\\2\\0\end{bmatrix}$. Inconclusive, he switched to the transformation notation by explicitly writing $\begin{cases}T\mathbf{v}_1=0\mathbf{v}_1+0\mathbf{v}_2+0\mathbf{v}_3\\T\mathbf{v}_2=0\mathbf{v}_1+2\mathbf{v}_2+0\mathbf{v}_3\\T\mathbf{v}_3=0\mathbf{v}_1+0\mathbf{v}_2+2\mathbf{v}_3\end{cases}$. This way of writing enabled him to obtain the diagonal matrix $A'=\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}$ while explaining: "*now it looks like a matrix.. a diagonal matrix with eigenvalues on the diagonal*." In theory, all students seemed to accept the possibility of having 0 as an eigenvalue, however, when it came to visualize or algebraically work it out, difficulties arose, as also shown in the present report.

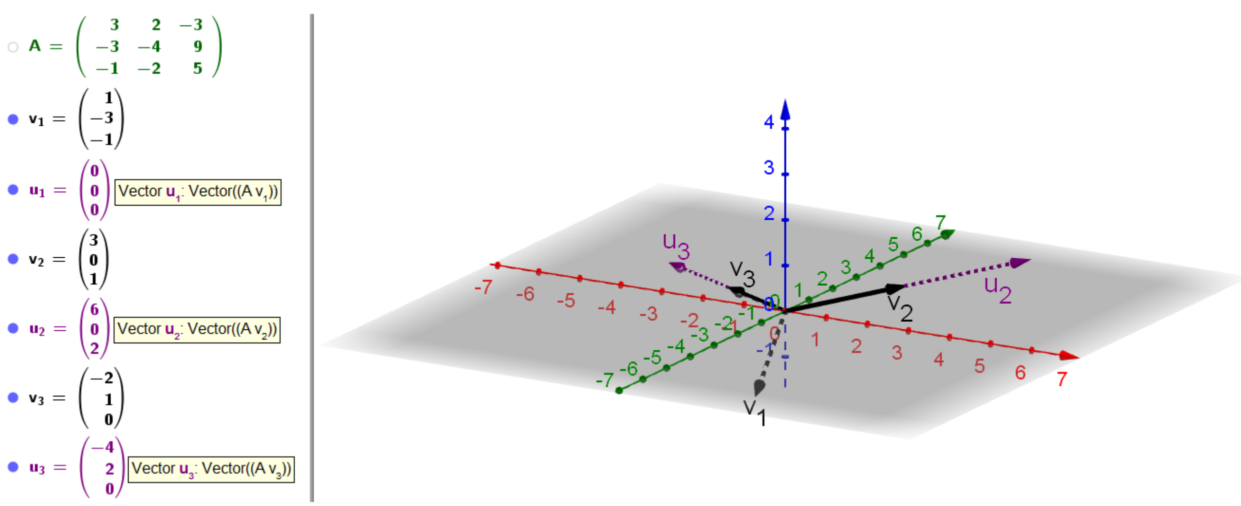

Figure 3 Typical student synthetic-geometric approach for Task 5.

Retrieving the negative eigenvalue from the matrix applied eigenvector. This difficulty occurred in the synthetic-geometric approach in the exploration of Task 3. Upon graphing the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and the *matrix-applied* basis vectors $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$, respectively, S9's first reaction was that that $\mathbf{v}_2$ and $\mathbf{v}_3$ were eigenvectors of A corresponding to the same positive eigenvalue $\lambda = 3$ (Figure 4a). Regarding $\mathbf{v}_1$, S9's first conjecture was that, rather than being a scalar multiple of $\mathbf{v}_1$, $A\mathbf{v}_1$ was perhaps a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. Upon using some techniques such as rotating the view, hiding the *xy*-plane, changing the scale by zooming in/out, hiding $\mathbf{v}_2$ and $\mathbf{v}_3$, etc., she ultimately came to the conclusion that $A\mathbf{v}_1$ was indeed "*three times* $\mathbf{v}_1$ *but pointing in opposite directions so the eigenvalue must be -3 and not 3* (Figure 4b)."

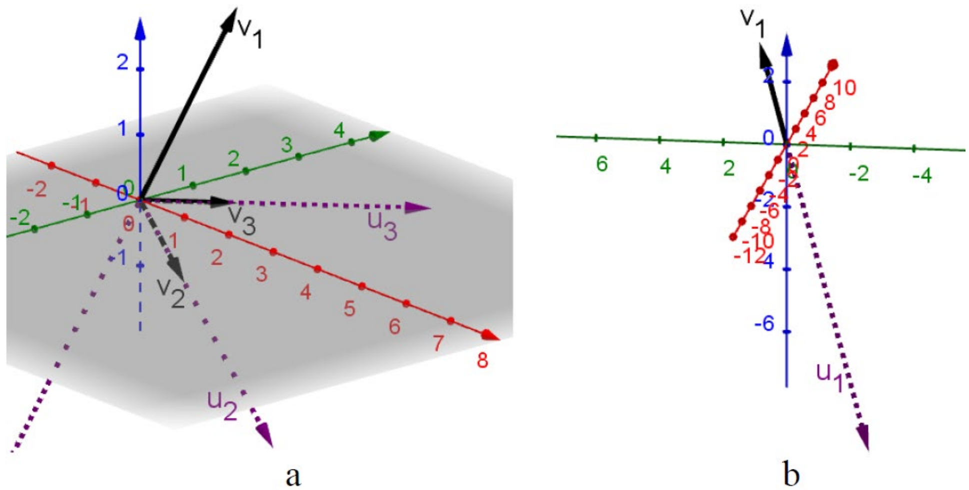

Figure 4 S9's synthetic-geometric approach for Task 3.

Issues within the analytic-arithmetic mode. S10, who embraced the analytic-arithmetic mode in Task 2, used a combination of *representation approach* and *object approach* by explicitly writing $\begin{cases} A\mathbf{v}_1 = \cdots = -2\mathbf{w}_1 + 10\mathbf{w}_2 - 10\mathbf{w}_3 \\ A\mathbf{v}_2 = \cdots = -6\mathbf{w}_1 + 2\mathbf{w}_2 + 2\mathbf{w}_3 \\ A\mathbf{v}_3 = \cdots = -3\mathbf{w}_1 + 0\mathbf{w}_2 + 2\mathbf{w}_3 \end{cases}$ (where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ denoted standard basis vectors for S10) which enabled him propose an incorrect matrix A'. Only after switching to the analytic-structural mode was he able to obtain the correct diagonal matrix representation as a result of computing $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (in a manner similar to S6 and S7's approaches described above but without using MATLAB). It can be postulated that S10's willingness to make use of the standard basis vectors within the analytic-mode is perhaps what caused the emergence of an incorrect matrix A'.

The issue was not the fact that she managed to obtain the correct answer at the end; the issue was, rather, that it took S11 quite longtime to solve three sets of systems of equations. S11, who embraced the analytic-arithmetic mode in Task 3, began with the *representation approach* by explicitly writing $\begin{cases} A\mathbf{v}_1 = \cdots = -2\mathbf{w}_1 + 10\mathbf{w}_2 - 10\mathbf{w}_3 \\ A\mathbf{v}_2 = \cdots = -6\mathbf{w}_1 + 2\mathbf{w}_2 + 2\mathbf{w}_3 \\ A\mathbf{v}_3 = \cdots = -3\mathbf{w}_1 + 0\mathbf{w}_2 + 2\mathbf{w}_3 \end{cases}$. However, she failed to see the eigenvalue-eigenvector relationships, which led her to write and solve three sets of systems of linear equations (each with three unknowns and three equations).

Issues within the analytic-structural mode. S5, who embraced the analytic-structural mode in Task 3, began by defining the inverse of the transition (change of basis) matrix as $P^{-1} = \beta'$ (as opposed to the others who defined the transition matrix as $P = \beta'$). This way, she obtained $A' =$

$P^{-1}AP = \begin{bmatrix} 13 & -20 & 0 \\ 8 & -13 & 0 \\ 4 & -8 & 3 \end{bmatrix}$. Upon the interviewer's probing how she knew this was the correct matrix representation for T, she switched to the analytic-arithmetic mode (based on pure *representation approach* unlike S10 who blended *representation approach* and *object approach*) and obtained another incorrect matrix for T due to the involvement of the standard basis vectors like S10. Inconclusive, she felt the need to switch to the synthetic-geometric approach via which she ultimately was able to obtain the correct matrix representation as $A' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Conclusion and Discussion

This report delved into mathematics students' interpretations of a classical problem in linear algebra, diagonal matrix representations of linear transformations with respect to eigenbases. All three modes of description in linear algebra appeared to have been adopted by the students; in some cases, students felt the strong need to switch between modes or utilize multiple modes of description. All students insistently questioned the meaning of having an eigenbasis such that the matrix representation for T relative to the given eigenbasis is diagonal. Using these multiple modes of description also enabled students acknowledge the importance of adopting a nonstandard basis with respect to which certain linear transformations revealed diagonal matrix representations. Some difficulties arose within each mode of description; notwithstanding, students addressed these difficulties by (i) paying close attention to the diagonalization structure; (ii) delving into the fundamental linear algebraic structures within and between modes description; and (iii) pondering deeply upon the desirableness of the diagonal matrix representation of the linear transformation under consideration.

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