The role of derivatives across multivariable and vector calculus instruction using multivariational reasoning and students' interpretations of matrix equations

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Multivariable and vector calculus (MVC) encompasses a wide array of different types of functions and the extension of numerous ideas from univariable calculus to MVC contexts. In this paper, I reflect on some of the more recent work on student thinking and learning of derivatives in MVC settings by leveraging some of the recent work on students' multivariational reasoning and student thinking and learning of linear algebra topics. I conclude with examples of ways to conceptualize derivatives as linear transformations to support students in progressing towards a unified notion of the derivative concept throughout MVC instruction.

Keywords: student thinking, multivariable calculus, derivatives, partial derivatives

Students in the United States are first introduced to the notions of multivariable and vector-valued functions in a multivariable and vector calculus (MVC) course. As a lead up to studying vector-valued functions the course introduces vectors, vector operations, graphing in three-dimensional coordinate systems, partial and directional derivatives, multiple integration, line and surface integrals, vector fields, the gradient operator, and the fundamental theorems of vector calculus (Stewart, 2011). Each of these concepts is important, but I am concerned that students perceive MVC as a collection of disconnected ideas and shortcuts (Harel, 2021), as opposed to an interconnected web of meanings. I am curious about ways students' understandings of MVC ideas are interconnected and how they extend their understanding of rate of change and accumulation functions (P. W. Thompson et al., 2019) to MVC contexts. In this paper, I want to reflect on ways to use current research in mathematics education to draw attention to how nuanced students' ways of thinking about MVC topics can be. To demonstrate my intention, I isolate my focus to the various uses of partial derivatives detailed across recent math education literature and reflect on how to make connections between students' reasoning about partial derivatives and their meanings for their symbolizations of linear approximations for different functions. I draw on co-variational and multivariational reasoning (Thompson & Carlson, 2017; Jones, 2022), in relation to an altered version of a theoretical framework from linear algebra (Andrews-Larson & Zandieh, 2013; Zandieh & Andrews-Larson, 2019).

A Review of the Literature on the Teaching and Learning of Partial Derivatives

Over the last five years, there has been an increase in mathematics education research investigating students' understandings and ways of thinking about multivariable calculus topics. These topics include: multiple integration (Jones & Dorko, 2015), ways of thinking about scalar and vector line integrals (Jones, 2020), students' treatments and conversions of graphical registers for gradient vectors (Moreno-Arotzena et al., 2021), manipulatives for supporting students in graphing surfaces (Kang et al., 2020; Wangberg, 2020; Wangberg et al., 2022), students' covariational reasoning about partial derivatives (Mhkatshwa, 2021), and students' multivariational reasoning (Jones, 2018, 2022). The various ways that partial derivatives are seemingly used by students and researchers are described in Fig. 1. Each use of partial derivative described in Fig. 1 is nuanced, but heavily emphasizes contexts of functions of two variables.

Interpretation	Description of interpretation	Related articles
Partial derivatives as slopes of tangent lines to traces	Intersect the graph of the function $z = f(x, y)$ with a plane $x = a$. Find the coordinate triple $(a, b, f(a, b))$ and draw the tangent line in the y direction contained in the plane $x = a$ at the given point. The slope of that tangent line is the value of the partial derivative of f with respect to y $f_y(a, b)$.	(Martínez-Planell et al., 2017)
Directional derivatives as slopes of a tangent plane in a particular direction	Intersect the graph of the function $z = f(x, y)$ with the plane $z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$ at the coordinate triple $(a, b, f(a, b))$. The change in depth (variation in z) measured with respect to the horizontal displacement (the length of the direction vector $\vec{u} = \langle \Delta x, \Delta y \rangle$ which is parallel to the xy plane) is the value of the directional derivative $D_{\hat{u}} f(a, b)$.	(Wangberg et al., 2022) (Martínez-Planell et al., 2017; McGee & Moore-Russo, 2015)
Partial derivatives as rate of change functions	Given an amount function $z = f(x, y)$ the rate of change of z with respect to y as y varies through a small-enough interval containing $y = b$, mentally holding the value of x fixed at $x = a$, is the value of the partial derivative $f_y(a, b)$.	(Mhkatshwa, 2021)
Partial derivatives as components of the gradient vector	Given a multivariable function $z = f(x, y)$, the gradient is a vector pointing in the direction of the greatest directional derivative $\|\|\vec{\nabla} f(a, b)\|\| = D_{\hat{u}} f(a, b)$ whose tail is located on the contour plot of f at (a, b). The x component of the gradient is the partial derivative of z with respect to x and the y component of the gradient is the partial derivative of z with respect to y so that $\vec{\nabla} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$.	(Moreno-Arotzena et al., 2021)
Partial derivatives as entries in the Jacobian matrix	Given a function $f: \mathbb{R}^2 \to \mathbb{R}^2$, the linear transformation, represented by the Jacobian matrix J, produces the best, local, linear approximation for f when $x_1 = a$ and $x_2 = b$. The components of matrix J are all of the first-order partial derivatives of the components of a given function, where each row is the transpose of the gradient of each component function.	(Harel, 2021)

Fig. 1: Interpretations of partial derivatives across recent math education research literature

Some of the earlier mathematics education research on MVC instructional design investigated students' responses to tasks within cycles of a Genetic Decomposition from APOS (action, process, object, scheme) theory which also leveraged Duval's theory of semiotic registers (Trigueros Gaisman & Martínez-Planell, 2013). The Genetic Decomposition for teaching partial derivatives of two-variable functions emerged from characterizing in-class activities for teaching two variable functions (Martínez-Planell & Trigueros Gaisman, 2019) and partial derivatives and directional derivatives through explicit lecture on 3D slopes (Martínez-Planell et al., 2017). McGee's tactile graphing manipulatives reinforced the notion that explicit attention to 3D slopes may provide students with a way to measure amounts of vertical change in terms of an amount of horizontal change, corresponding to the value of the directional derivative of a given two-variable function (McGee et al., 2012, 2015; McGee & Martínez-Planell, 2014; McGee & Moore-Russo, 2015). Students reasoning about 3D slopes (in the x and y direction respectively) and geometric conceptions of constant rate of change (as slopes) provided them with ways to reason about the values of partial derivatives graphically in 3D space. The research detailing the instructional implementations of the 3D acrylic surfaces of Wangberg and colleagues' also supported the notion that students reasoning about multivariable rates of change in graphical contexts require ways to envision measuring amounts of vertical in terms of the corresponding horizontal change to conceive of the values of directional derivatives as the slope of the surface in a given direction (Wangberg, 2020; Wangberg et al., 2022). Reasoning about surfaces as something to be traveled upon is extremely useful, however care should also be taken to ensure students make connections between tactile approaches to graphing surfaces and also reasoning about how the shape of the graph is a record of how the values of three quantities change together (Moore & Thompson, 2015; Weber & Thompson, 2014).

There are fewer research articles investigating how students conceive of vector-valued amount and rate functions. Jones investigated how students think about scalar and vector line integrals (Jones, 2020). Researchers have also investigated the ways students think about vectors and vector operations in the context of linear algebra (Lee, 2022), or physics (Dray & Manogue, 2006; Roche, 1997). However, there is a need for research experiments investigating how students think about and learn calculus ideas related to the derivatives of vector-valued functions, (i.e., divergence, curl, Green and Stokes' theorem, etc.). The research literature in physics education has also addressed ways to support students in using partial derivatives, and the relevant notation, to reconstruct Maxwell's equations of thermodynamics (Cannon, 2004; J. R. Thompson, 2006; J. R. Thompson et al., 2012). Some physics education researchers have leveraged Zandieh's derivative framework to determine the efficacy of tasks and instructional interventions in thermodynamics courses (Bajracharya et al., 2019). Further, Roundy and colleagues discussed how instructional issues may emerge when different disciplinary practices impact students' conceptions of the use of partial derivatives (Roundy et al., 2015). For instance, mathematicians and physicists appear to leverage partial derivatives in different ways in their disciplines. It would be prudent to continue the dialogue between math and physics instructors at each institution to determine ways to support students in shifting or extending their understanding of partial derivatives flexibly across the STEM disciplines.

Theoretical perspective

Before elaborating on various interpretations and uses of partial derivatives for different function types, I am going to discuss students' conceptions of different types of functions. MVC courses encompass many different types of functions sorted into at least two groups: (1) functions with multiple independent quantities and (2) functions with multiple dependent quantities (including vector-valued outputs). When discussing functions with more than one independent quantity, I will say *multivariable* functions (i.e., $z = f(x, y)$ or $w = g(x, y, z)$ as opposed to *univariable* functions (i.e., $y = f(x)$). To distinguish between functions with multiple dependent quantities and a single dependent quantity I will say *vector-valued* and *scalar-valued* respectively. It is highly unlikely that most students have encountered multivariable functions, or the corresponding notation, by the time they enroll in MVC. Some students may have encountered vectors and vector operations in the context of physics or linear algebra (LA). In Calculus I and II, most students have encountered univariable function notation, input/output tables, and memorized information about the graphs of various function families, along with a thorough introduction to derivative and integral techniques. However, it appears less likely that most student have been supported in reflecting on the idea of univariable functions (across various representational contexts) as sets of invariant relationships relating the values of two quantities whose values vary in tandem with the realization that, at any moment, the value of the independent quantity determines exactly one value of the dependent quantity (P. W. Thompson et al., 2017; P. W. Thompson & Carlson, 2017).

The literature reporting on researchers' investigations and theories about student thinking related to the function concept encompasses several decades of experimental results and theory (Breidenbach et al., 1992; M. P. Carlson, 1998; Monk, 1994; Oehrtman et al., 2008; Sfard, 1992; Zandieh et al., 2017). While there have been some studies investigating student thinking about functions of two variables (Kabael, 2011; Martínez-Planell & Trigueros Gaisman, 2019; Yerushalmy, 1997), the math education theory of student thinking about multivariable and vector-valued functions is comparatively less developed (Rasmussen & Wawro, 2017). I am genuinely curious about how students make sense of the sheer number of different *types* of functions covered in a single semester of MVC instruction (i.e., multivariable functions, parametric equations, vector-valued functions, vector fields, linear and non-linear transformations), in relation to their understanding of calculus ideas related to each function type (e.g., the divergence and curl of multivariable vector-valued functions graphed as vector fields).

Mathematics education researchers might ask research questions related to the various ways students conceive of different types of functions (e.g., action vs. process views of functions of two variables, a structural vs. operational views of linear transformations, covariational or multivariational views of parametric functions). Students' conceptions of univariable functions likely impact their understanding of multivariable functions, and eventually more general mappings such as linear transformations (Trigueros Gaisman & Martínez-Planell, 2013; Yerushalmy, 1997; Zandieh et al., 2017). Addressing the ways students think about and generalize their understanding of the function concept across their undergraduate mathematics course sequence is an important consideration for research and instructional design. Zandieh, Ellis, and Rasmussen (2017) analyzed students' conceptions the function concept cutting across the students experiences from high school algebra to linear transformations in linear algebra. I am curious about how students' notion of function, unified or otherwise, interacts with their conception of derivatives across various MVC contexts. Investigating students' *stable ways of thinking* about functions (P. W. Thompson et al., 2014) seems like a necessary first step to investigating how students conceptualize multivariable rate functions and partial derivatives more generally. Leveraging theories to investigate how students think, generalize, and symbolize their conceptions of functions, and their derivatives, is also worthwhile and relevant to the current trajectory of mathematics education research.

To that end, Jones recently expanded his multivariational framework (Jones, 2018, 2022) based on some of the work emerging from Carlson et al.'s (2002) covariational reasoning framework and Thompson and Carlson's (2017) variation and extended covariation frameworks. Jones described three types of multivariation as well as a framework describing students' multivariational reasoning (MR) mental actions. Jones defined three types of multivariation: (1) independent multivariation (i.e., functions of the form $z = f(x, y)$), (2) dependent multivariation (i.e., situations like the ideal gas law $PV = kT$), and (3) nested multivariation (i.e., composite, or parametric functions $y = f(x)$ where $x = g(t)$). Jones' multivariation and MR framework seem viable for building models of student thinking to determine how students engage in quantitative reasoning (P. W. Thompson, 1993, 2022) about multivariable and vector-valued functions. Further, modeling student thinking to determine how they coordinate the values of multiple quantities changing together is a viable lens for investigating how students conceptualize multivariable rates of change and, eventually, multivariable rate functions.

For example, suppose a student possesses a process view of the function $f(x, y) = xy^2 - xy$ where $z = f(x, y)$ (independent multivariation). The student mentally fixes the value of $x = a$, through decomposing the situation into *isolated covariations*. By considering how the value of z

co-varies with y as the value of y varies from -2 to 2 (holding the value of x fixed), the student can rely on previous ways of thinking about graphs of univariable functions. The student coordinates their mental image of the 2D graphs of $z = ay^2 - ay = ay(y - 1)$ for particular values of a with images of the various traces situated in the planes $x = a$ pictured in $\mathbb{R}^3$. The student may reconceive of the situation and anticipate the shape of the surface as an emergent process through mentally coordinating the changing shape of the traces as x varies from -2 to 2. This is exactly the hypothetical learning trajectory (HLT) proposed by Thompson and Weber via the *sweeping out* metaphor (Weber & Thompson, 2014).

Mentally fixing the value of $x = a$ corresponds to the *reducing into isolated covariations* MR mental action and sequentially reconceiving of x as a varying quantity corresponds to the *coordination of multiple simultaneous changes* MR mental action, and perhaps, eventually, *smooth continuous multivariation*. This series of Jones' set of MR mental actions highlights one of the ways a student may come to graphically quantify the partial derivative of the function $f(x, y) = xy^2 - xy$ with respect to y, holding the value of x fixed at $x = a$, by associating their images of each trace of the surface which emerged through their initial conception of how the values of y and z *co*-varied together. The linear approximation for this particular trace (corresponding to the intersection of the graph of $z = f(x, y)$ and the plane $x = a$) emerges by assuming a constant rate of change (specifically $f_y(a, b)$) relating amounts of variation in the value of y away from $y = b$, and the corresponding amount of variation in the value of z emerges from pretending there is a constant rate of change relating amounts of variation in the values of y and z, i.e., $\Delta z \cong f_y(a, y)\Delta y$ or $z - f(a, b) \cong f_y(a, b)(y - b)$. Providing students with the opportunity to conceptualize partial derivatives as imagined constant rates of change due to variation in the value of y, holding the value of x fixed, may undergird why explicit discussions 3D slopes have proven helpful to students when reasoning about the values of partial derivatives as the slopes of tangent lines to the trace of f (Martínez-Planell et al., 2017).

Zandieh's (1997; 2000) derivative framework demonstrated how nuanced students' conceptions of derivatives can be. Each of the three process-object layers (ratio, limit, and function) encapsulates the *reification* of a particular process into a more static, *structural* conception of that layer (Sfard, 1992). The columns of Zandieh's framework encompass various contexts in which the framework can be used to depict the various aspects of a student's conception of derivatives. As Zandieh (1997; 2000) noted in the articulation of her derivative framework, each of the process-object pairs (the rows of the framework) correspond to a particular component of the derivative concept. The ratio, limit, and function layer may have their uses in particular contexts, but each aspect of the derivative concept is important to the overall conceptual structure. Students may have conceived of aspects of the derivative concept as contextual, or comprising the entire concept (Zandieh & Knapp, 2006). For example, in graphical settings, students may claim derivatives are slopes of tangent lines, but in a related rates task, a student may claim derivatives are just rate of change functions, without recognizing an implicit graphical connection to their previous statements about slopes of tangent lines. Investigating the ways students extend their understanding of derivatives to multivariable contexts is non-trivial. However, Zandieh's derivative framework is viable for articulating some of the nuanced aspects of the derivative concept in MVC. One such aspect of derivatives in MVC contexts, that is rarely discussed, is the conception of *derivatives as linear transformations*, as represented by the Jacobian matrix (Gravesen et al., 2017; Harel, 2021).

Zandieh, Ellis, and Rasmussen addressed the ways that students interpret linear transformations in relation to their high-school conceptions of function (Zandieh et al., 2017).

Zandieh et al. (2017) noted that students' concept images of function (from $\mathbb{R} \to \mathbb{R}$) may be separate from their conception of linear transformations as matrix equations. However, as students progress through the semester, their concept images become more interwoven to shift towards a unified notion of the function concept. I hypothesize that direct discussions of linear transformations are necessary to support students in extending their understanding of derivatives, more generally, towards a unified notion of the derivative concept. Zandieh and Andrews-Larsen discussed the various ways students interpret matrix equations (Larson & Zandieh, 2013), and extended their framework to include more nuanced interpretations of systems of linear equations and matrix equations (Zandieh & Andrews-Larson, 2019). I have outlined theoretical students' interpretations of the various roles that partial derivatives play in producing linear approximations for two types of functions in Fig. 2. Again, the various types of derivatives present in MVC may perpetuate students' apparent conception that MVC topics are seemingly disjoint and highly contextual. I hypothesize that centering the idea of derivative as a linear function, or transformation, which produces the best local, linear approximation of *any* type of function will help support students in conceiving of a more unified notion of derivative and thus leading towards coherence of MVC instruction in the minds of students.

Interpretation of $J_f\Delta\vec{x}$	**LT example**: $f: \mathbb{R}^2 \to \mathbb{R}^2$ $y_1 = P(x_1, x_2)$ $y_2 = Q(x_1, x_2)$	**MVC example**: $f: \mathbb{R}^2 \to \mathbb{R}$ $z = f(x, y)$	**Verbal description of each interpretation**
Linear Combination Interpretation (**LC**)	***Linear combination of amounts of change:*** $\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix} \cong \begin{bmatrix}\frac{\partial y_1}{\partial x_1}\\ \frac{\partial y_2}{\partial x_1}\end{bmatrix}\Delta x_1 + \begin{bmatrix}\frac{\partial y_1}{\partial x_2}\\ \frac{\partial y_2}{\partial x_2}\end{bmatrix}\Delta x_2$ $\Delta\vec{y} \cong \frac{\partial\vec{y}}{\partial x_1}\Delta x_1 + \frac{\partial\vec{y}}{\partial x_2}\Delta x_2$	***Coordination of amounts of change***: $\Delta z_x \cong f_x(a, b)\Delta x$ $\Delta z_y \cong f_y(a, b)\Delta y$ $\Delta z \cong \Delta z_x + \Delta z_y$	Approximation emerging from coordinating the amount of variation in the value of the vector or scalar-valued dependent quantity due to variations in the value of each independent quantity separately.
System of Equations Interpretation (**SE**)	***System of linear approximations***: $\Delta y_1 \cong \frac{\partial y_1}{\partial x_1}\Delta x_1 + \frac{\partial y_1}{\partial x_2}\Delta x_2$ $\Delta y_2 \cong \frac{\partial y_2}{\partial x_1}\Delta x_1 + \frac{\partial y_2}{\partial x_2}\Delta x_2$	***Linear approximation***: $\Delta z \cong \frac{\partial f(a, b)}{\partial x}\Delta x + \frac{\partial f(a, b)}{\partial y}\Delta y$ $z \cong \frac{\partial z}{\partial x}(x - a) + \frac{\partial z}{\partial y}(y - b) + f(a, b)$	Approximation for each dependent quantity (as a component of a vector-valued output) due to simultaneous variation in the value of each independent quantity.
Dot product interpretation (**DotP**)	***System of dot products***: $\Delta y_1 \cong \vec{\nabla}P(a, b) \cdot \begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$ $\Delta y_2 \cong \vec{\nabla}Q(a, b) \cdot \begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$	***Dot product***: $\Delta z \cong \vec{\nabla}f(a, b) \cdot \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$	Approximation for each dependent quantity as emergent from a geometric relationship as projection or based on the angle between the gradient vector and a direction vector.
Transformation interpretation (**LT**)	***Approximating a non-linear transformation***: $\Delta\vec{y} \cong J_f\Delta\vec{x}$ $\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix} \cong \begin{bmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$	***Inner product***: $\Delta z \cong \vec{\nabla}^T f(a, b)\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$ $\Delta z \cong \begin{bmatrix}\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y}\end{bmatrix}\begin{bmatrix}dx\\ dy\end{bmatrix}$	Approximation of the output space via the linear transformation represented by the Jacobian matrix.

Fig. 2: Theoretical student interpretations of linear approximations altered from Zandieh & Andrews-Larson (2019). The geometric context column has been omitted due to space limitations.

Based on some of the uses of partial derivatives pictured in Fig. 1 and the relevant math education literature on student thinking about partial derivatives, I altered Zandieh and Andrews-

Larson's framework for students' interpretations of matrix equations to anticipate various ways students could interpret and symbolize derivatives in the context of constructing linear approximations for different types of functions. The table in Fig. 2 could be used to anticipate students' conceptions and symbolization for scalar and vector-valued functions with more than two independent or dependent quantities (e.g., multivariable, scalar-valued functions with three or four independent quantities). While the table in Fig. 2 only elaborates on linear approximations for functions involving independent multivariation, the interpretations can be applied to situations involving nested multivariation through the chain rule and conceptualizing products of matrices as compositions of linear transformations (Harel, 2021).

The MR mental action of *reducing to isolated covariations* seems necessary to conceive of linear approximations while coordinating the amount of variation in the value of the output vector or scalar-valued dependent quantity (z) due to separate variations in the value of each pair of independent quantities (x_1 and y_1 and x and y respectively). Coordinating the amount of total variation, or at least an approximation of this amount of variation, emerges from the *coordination of multiple simultaneous changes* MR mental action. The LC and SE interpretations more readily highlight the *reducing to isolated covariations* and *coordination of simultaneous changes mental actions* due to the emphasis on coordinating net changes in the values of the dependent quantities due to separate amounts of variation in the values of each independent quantity.

Conclusion

One way to continue to build on the current state of the mathematics education research literature on the teaching and learning of MVC topics is through coordinating research on student thinking about linear algebra concepts in conjunction with burgeoning theories about student thinking within the context of MVC instruction (Cobb, 2007). One approach is to leverage an altered version of Zandieh and Andrews-Larsen's (2019) framework elaborating students' interpretations of matrix equations in the context of constructing linear approximations for multivariable and vector-valued functions with Jones' (2022) elaboration of MR to anticipate ways students can interpret and symbolize derivatives in MVC. The ways in which students conceive of multivariable and vector-valued rate of change functions necessarily depends on their conceptions of amount functions. Future research studies should seek to coordinate students' symbolization activity with their associated meaning in graphical and non-graphical contexts (Tyburski et al., 2021). Triangulating students' meanings and stable understandings of vector calculus ideas will help to extend the research field's understanding of how students learn and interpret vector calculus ideas and notation.

In this manuscript, I have summarized the recent work on the ways students use partial derivatives in MVC. It should be restated that there are different *types of functions* present in a MVC course, but there are also numerous ways students could conceptualize them; as one example is the diverse approaches to coordinating variation in the values of numerous quantities. As Zandieh, Rasmussen, and Ellis noted, students' conceptions of the function concept may vary widely depending on the context. My hope is to gain insights into the nuanced ways students interpret, use, and symbolize partial derivatives when creating linear approximations.

Future studies could extend the theory I have presented here by applying it to students' written work in MVC contexts. This report is largely theoretical and based almost entirely on my experiences working with students enrolled in a conceptually oriented MVC course for two academic years in conjunction with my research with the IOLA research group.

Bibliography

Andrews-Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation Ax=b. *For the Learning of Mathematics*, *33*(2), 11–17.

Bajracharya, R. R., Emigh, P. J., & Manogue, C. A. (2019). Students' strategies for solving a multirepresentational partial derivative problem in thermodynamics. *Physical Review Physics Education Research*, *15*(2), 020124. https://doi.org/10.1103/PhysRevPhysEducRes.15.020124

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Cannon, J. W. (2004). Connecting thermodynamics to students' calculus. *American Journal of Physics*, *72*(6), 753–757. https://doi.org/10.1119/1.1648327

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education*, *33*(5), 352. https://doi.org/10.2307/4149958

Carlson, M. P. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. In A. H. Schoenfeld, J. Kaput, and E. Dubinsky (Eds.). *Research in Collegiate Mathematics Education III*, *3*, 114–162.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 3–38). Information Age Pub.

Dray, T., & Manogue, C. A. (2006). The geometry of the dot and cross products. *Journal of Online Mathematics and Its Applications*, *6*, 1–13.

Gravesen, K. F., Grønbæk, N., & Winsløw, C. (2017). Task Design for Students' Work with Basic Theory in Analysis: The Cases of Multidimensional Differentiability and Curve Integrals. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 9–33. https://doi.org/10.1007/s40753-016-0036-z

Harel, G. (2021). The learning and teaching of multivariable calculus: A DNR perspective. *ZDM Mathematics Education*. https://doi-org.ezproxy1.lib.asu.edu/10.1007/s11858-021-01223-8

Jones, S. R. (2018). Building on Covariation: Making Explicit Four Types of "Multivariation." *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 1110–1118.

Jones, S. R. (2020). Scalar and vector line integrals: A conceptual analysis and an initial investigation of student understanding. *The Journal of Mathematical Behavior*, *59*(2), 1–18.

Jones, S. R. (2022). Multivariation and students' multivariational reasoning. *Journal of Mathematical Behavior*, *67*, 1–18. https://doi.org/10.1016/j.jmathb.2022.100991

Jones, S. R., & Dorko, A. (2015). Students' understanding of multivariate integrals and how they may be generalized from single integral conceptions. *The Journal of Mathematical Behavior*, *40*, 154–170.

Kabael, T. (2011). Generalizing Single Variable Functions to Two-variable Functions, Function Machine and APOS. *Educational Sciences: Theory & Practice*, *11*(1), 484–499.

Kang, K., Kushnarev, S., Wei Pin, W., Ortiz, O., & Chen Shihang, J. (2020). Impact of Virtual Reality on the Visualization of Partial Derivatives in a Multivariable Calculus Class. *IEEE Access*, *8*, 58940–58947. https://doi.org/10.1109/ACCESS.2020.2982972

Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation Ax=b. *For the Learning of Mathematics*, *33*(2), 11–17.
Lee, I. (2022). What is a vector to students? *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*, 1061–1066.
Martínez-Planell, R., & Trigueros Gaisman, M. (2019). Using cycles of research in APOS: The case of functions of two variables. *The Journal of Mathematical Behavior*, *55*, 100687.
Martínez-Planell, R., Trigueros Gaisman, M., & McGee, D. (2017). Students' understanding of the relation between tangent plane and directional derivatives of functions of two variables. *The Journal of Mathematical Behavior*, *46*, 13–41.
McGee, D., & Martínez-Planell, R. (2014). A study of semiotic registers in the development of the definite integral of functions of two and three variables. *International Journal of Science and Mathematics Education*, *12*(4), 883–916.
McGee, D., & Moore-Russo, D. (2015). Impact of Explicit Presentation of Slopes in Three Dimensions on Students' Understanding of Derivatives in Multivariable Calculus. *International Journal of Science and Mathematics Education*, *13*(2), S357–S384.
McGee, D., Moore-Russo, D., Ebersole, D., Lomen, D. O., & Quintero, M. M. (2012). Visualizing Three-Dimensional Calculus Concepts: The Study of a Manipulative's Effectiveness. *PRIMUS*, *22*(4), 265–283.
McGee, D., Moore-Russo, D., & Martínez-Planell, R. (2015). Making Implicit Multivariable Calculus Representations Explicit: A Clinical Study. *PRIMUS*, *25*(6), 529–541.
Mhkatshwa, T. P. (2021). A quantitative and covariational reasoning investigation of students' interpretations of partial derivatives in different contexts. *International Journal of Mathematical Education in Science and Technology*, 1–23.
Monk, G. S. (1994). Students' Understanding of Functions in Calculus Courses. *Humanistic Mathematics Network Journal*, *9*, 21–27.
Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. *Proceedings of the Eighteenth Annual Conference on Research in Undergraduate Mathematics Education*, 782–789.
Moreno-Arotzena, O., Pombar-Hospitaler, I., & Ignacio Barragués, J. (2021). University student understanding of the gradient of a function of two variables: An approach from the perspective of the theory of semiotic representation registers. *Educational Studies in Mathematics*, *106*, 65–89. https://doi.org/10.1007/s10649-020-09994-9
Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational Reasoning Abilities that Promote Coherence in Students' Function Understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection* (pp. 27–42). The Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.004
Rasmussen, C., & Wawro, M. (2017). Post-Calculus Research in Undergraduate Mathematics Education. In *Compendium for Research in Mathematics Education* (pp. 551–579). National Council of Teachers of Mathematics.
Roche, J. (1997). Introducing vectors. *Physics Education*, *32*, 339–344.
Roundy, D., Weber, E., Dray, T., Bajracharya, R. R., Dorko, A., Smith, E. M., & Manogue, C. A. (2015). Experts' understanding of partial derivatives using the partial derivative machine. *Physical Review Special Topics - Physics Education Research*, *11*(2), 020126. https://doi.org/10.1103/PhysRevSTPER.11.020126

Sfard, A. (1992). Operational Origins of Mathematical Objects and the Quandary of Reification-The Case of Function. In *The concept of function: Aspects of epistimology and pedagogy* (In G. Harel&E. Dubinsky (Eds.), pp. 59–84). Mathematical Association of America.

Stewart, J. (2011). *Multivariable calculus* (7th ed.). Brookes/Cole Cengage Learning.

Thompson, J. R. (2006). Assessing Student Understanding of Partial Derivatives in Thermodynamics. *AIP Conference Proceedings*, *818*, 77–80. https://doi.org/10.1063/1.2177027

Thompson, J. R., Manogue, C. A., Roundy, D. J., Mountcastle, D. B., Rebello, N. S., Engelhardt, P. V., & Singh, C. (2012). *Representations of partial derivatives in thermodynamics*. 85–88. https://doi.org/10.1063/1.3680000

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *25*(3), 165–208. https://doi.org/10.1007/BF01273861

Thompson, P. W. (2022). Quantitative reasoning as an educational lens. *Quantitative Reasoning in Mathematics and Science Education*.

Thompson, P. W., Ashbrook, M., & Milner, F. (2019). *Calculus, Newton, Leibniz, and Robinson meet technology*. http://patthompson.net/ThompsonCalc/

Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically. In *Compendium for Research In Mathematics Education* (In J. Cai (Ed.), pp. 421–456). National Council of Teachers of Mathematics.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes For Thinking With Magnitudes: A Hypothesis About Foundational Reasoning Abilities In Algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.). *Epistemic Algebra Students: Emerging Models of Students' Algebraic Knowing*, *4*, 1–24.

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior*, *48*, 95–111. https://doi.org/10.1016/j.jmathb.2017.08.001

Trigueros Gaisman, M., & Martínez-Planell, R. (2013). Contribution of the Dialogue Between Two Theories to the Study of Two Variable Functions. Martinez, M. & Castro Superfine, A (Eds.). *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 529–533.

Tyburski, B., Drimilla, J., Byerly, C., Boyce, S., Grabhorn, J., & Moore, K. C. (2021). From theory to methodology: Guidance for analyzing students' covariational reasoning. *Proceedings of the 43rd Annual Meeting of PME-NA*, 1839–1848.

Wangberg, A. (2020). Student Reasoning with Graphs, Contour Maps, and Rate of Change for Multivariable Functions. *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*, 631–637.

Wangberg, A., Gire, E., & Dray, T. (2022). Measuring the derivative using surfaces. *Teaching Mathematics and Its Applications: An International Journal of the IMA*, 110–124. https://doi.org/10.1093/teamat/hrab030

Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics*, *87*(1), 67–85. https://doi.org/10.1007/s10649-014-9548-0

Yerushalmy, M. (1997). Designing Representations: Reasoning about Functions of Two Variables. *Journal for Research in Mathematics Education*, *28*(4), 431–466.

Zandieh, M., & Andrews-Larson, C. (2019). Symbolizing while solving linear systems. *ZDM Mathematics Education*, *51*, 1183–1197. https://doi.org/10.1007/s11858-019-01083-3

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, *95*(1), 21–38.

Zandieh, M., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *Journal of Mathematical Behavior*, *25*, 1–17.