Affordances and Challenges Associated with Euler Diagrams as Representations of Logical Implications

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Research in mathematics education has begun to capture students' epistemic experiences with logic and proof in general, and with logical implications in particular (e.g., Antonini, 2004; Dawkins, 2017; Durand-Guerrier, 2003; Durand-Guerrier et al., 2012; Epp, 2003; Roh & Lee, 2018; Stylianides et al., 2004; Yopp, 2017). However, much is still unknown about how instruction can support student reasoning and learning about logical implication. Some researchers have explored the use of Euler diagrams for representing relations between sets and, in particular, truth sets of open statements (e.g., Bronkorst et al., 2021; Dawkins & Roh, 2022; Hub & Dawkins, 2018). In our ongoing research (the Proofs Project), we use Euler diagrams in introduction-to-proofs courses as one way to represent logical implications and their transformations. We have been conducting clinical interviews with students in which we investigate their conceptual mappings between logical implications and Euler diagrams. We share preliminary findings from our initial analyses of interview data to address the following question: *What affordances and challenges do Euler diagrams present as visual-spatial representations of logical statements?*

One affordance of Euler diagrams comes from their spatial nature: they provide a means to *visualize* relationships between two truth sets. One student (Carmen) explained, "I think with the if-then statements, it helps me to think of the Euler diagrams… and thinking of a circle within a circle." Another student (Kai) commented that, while he preferred to use symbolic logic over Euler diagrams, especially for representing more complex statements, he added, "Otherwise, I think [an Euler diagram] shows relationships very clearly if you have the initial setup correct."

The spatial nature of Euler diagrams also presents particular challenges for students. For instance, ideally, the statement $P \rightarrow Q$ is mapped to an Euler diagram in which the truth set of P is a subset of the truth set of Q. Some students in our study represented this implication with P as a *superset* of Q. As Carmen explained, "The P engulfs the entire space of Q, so that if I have the statement P, then I'm definitely going to have the statement Q." Additionally, for contrapositive statements (e.g., $\sim Q \rightarrow \sim P$), visualizing the truth set of $\sim Q$ as a subset of $\sim P$ in a 2D Euler diagram in which the truth sets of P and Q are depicted first may be difficult as it requires first visualizing the complements of the truth sets for P and Q as closed regions in 2D Euclidean space. Furthermore, the containment of $\sim Q$ within $\sim P$ may be obscured for students because the resulting depiction of $\sim Q \rightarrow \sim P$ is not homeomorphic to the original. We note that, ideally, Euler diagrams for two logically equivalent statements would be homeomorphic to each other. Because these spaces *are* homeomorphic when visualized on a sphere (rather than 2D Euclidean space), we discuss potential affordances and challenges of such a representation, and how Peircian diagrams on a sphere may provide alternative spatial representations (cf. Pietarinen, 2016).

Acknowledgments

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Purpose & Motivation

Goals of the Proofs Project

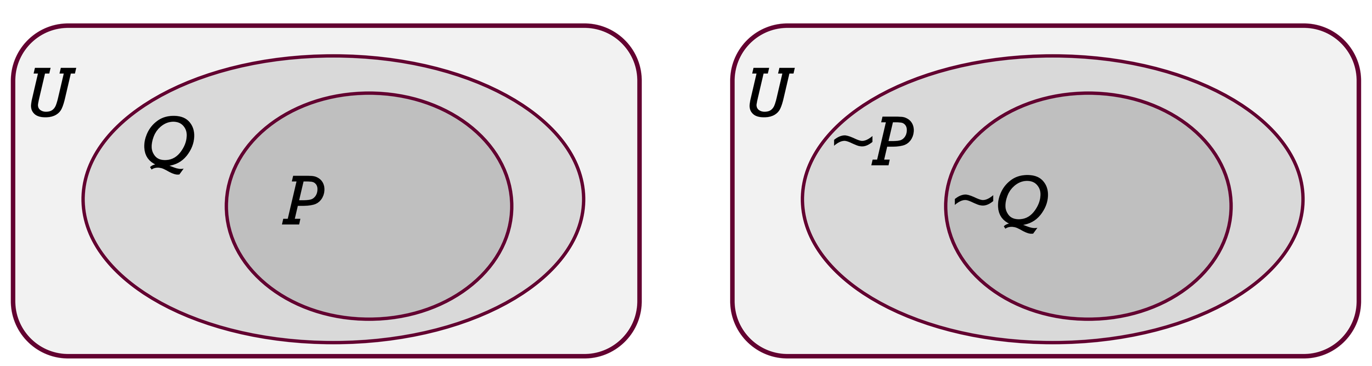
- Identify and address students' epistemological obstacles in introductory proofs courses
- Support teachers by suggesting instructional interactions that may provide students with opportunities to identify, reflect on, and overcome these obstacles

Research Questions Regarding Euler Diagrams

- How do students interpret, construct, and make sense of Euler diagrams as representations of logical implications?
- What epistemological affordances and challenges may be associated with this mode of representation?

What are Euler Diagrams?

- Visual-spatial representations showing relationships between sets
- Statement $p \to q$ represented as $P \subseteq Q$, where P and Q are the truth sets of statements p and q, respectively
- U represents the universal set, or the domain of objects under consideration



(a) (b)

Figure 1: Euler diagram representations of (a) $p \to q$ and (b) $\sim q \to \sim p$

Why Euler Diagrams?

- Hub and Dawkins (2018) suggest students can abstract the logic of conditionals by populating sets with familiar mathematical objects.
- Conditionals describe a relationship between properties. If students populate sets with those properties, they can represent the relationship spatially, as a relationship between sets.
- Dawkins and Roh (2021) have employed this instructional approach, relying heavily on the use of Euler diagrams, within their own research project.

Methods

- Classroom data from an introductory proofs course, taught in the Fall 2022 semester
- Three clinical interviews with each of four students in the course: (1) Logical Implication, (2) Quantification, and (3) Mathematical Induction and Stimulated Recall

Affordances

Visual Representation of an Abstract Relationship

- **Carmen:** "I think with the if-then statements, it helps me to think of the Euler diagram... and thinking of a circle within a circle."
- **Kai:** Stated that he prefers symbolic representations, but also said, "I think [an Euler diagram] shows relationships very clearly if you have the initial setup correct."
- Allows students to offload cognitive demands of maintaining relationships between logical structures by establishing a spatial representation

Visual Facilitation of Transformations

- Provide students with a visual aid in interpreting transformations of logical implications (e.g., negation)

Challenges

Different Interpretation of Subset Relation

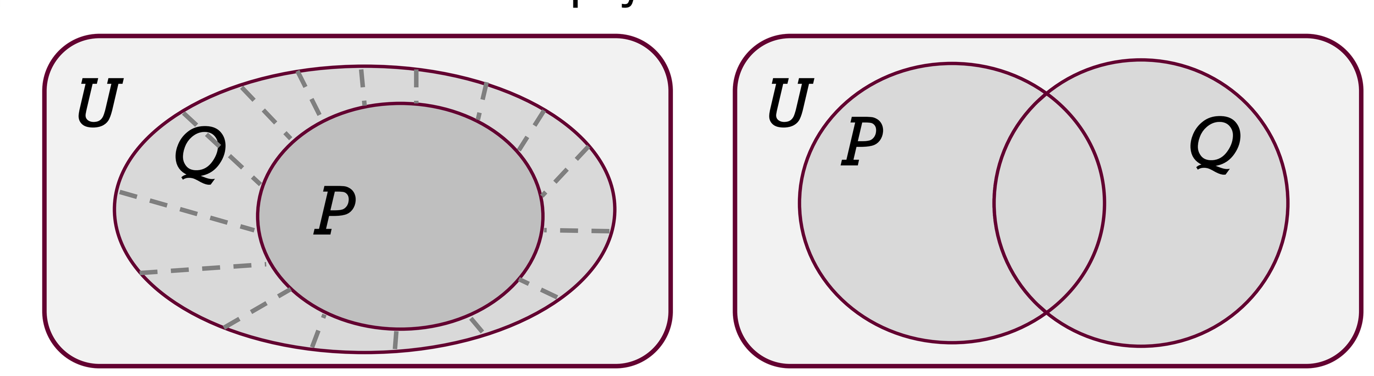
- **Carmen:** "So, if p, then q. The P engulfs the entire space of Q so that if I have the statement p, then I'm definitely going to have the statement q."
- Some students interpreted the containment of P in Q as suggesting that, if you have the space P (or statement p), then you have the space Q (or statement q)
- As **Kai** noted in excerpt above, Euler diagrams can be useful, but they require
  - structuring spatial representations of logical information in a meaningful way, along with
  - shared understanding of what set containment within an Euler diagram represents

Size Matters Not

- Qualitative nature of Euler diagrams may hide quantification from students
- Canonically, the size of a space within an Euler diagram is not an indicator of the cardinality of the set that it represents
- A circle may be used to represent a singleton set, whereas another circle (perhaps smaller than the original) could be used to represent an infinite collection

Empty or Not Empty?

- Generally, the existence of a closed space in an Euler diagram implies the space represents a non-empty set
- Not always the case: in the Euler diagram for $p \to q$, the space $Q - P$ may be empty if $q \to p$ also holds (see Fig. 2a), or the space P may be empty (vacuous case)
- By contrast, in the Euler diagram shown in Fig. 2b, $P \cap Q$ is presumed to be non-empty



(a) (b)

Figure 2: (a) $Q - P$ may be empty, but (b) $P \cap Q$ is assumed to be non-empty

Representing the Contrapositive

- Euler diagram for $p \to q$ not topologically equivalent to Euler diagram for $\sim q \to \sim p$ (compare Fig. 1a and Fig. 1b)
- One potential remedy: construct and transform Euler diagrams on a sphere (see Fig. 3)
- Starting with $P \subseteq Q$, dilate P and Q, reaching to the back side of the sphere, until only a small region not contained in Q is remaining, then invert relationship to show $\sim Q \subseteq \sim P$
- Unclear what additional obstacles this may introduce

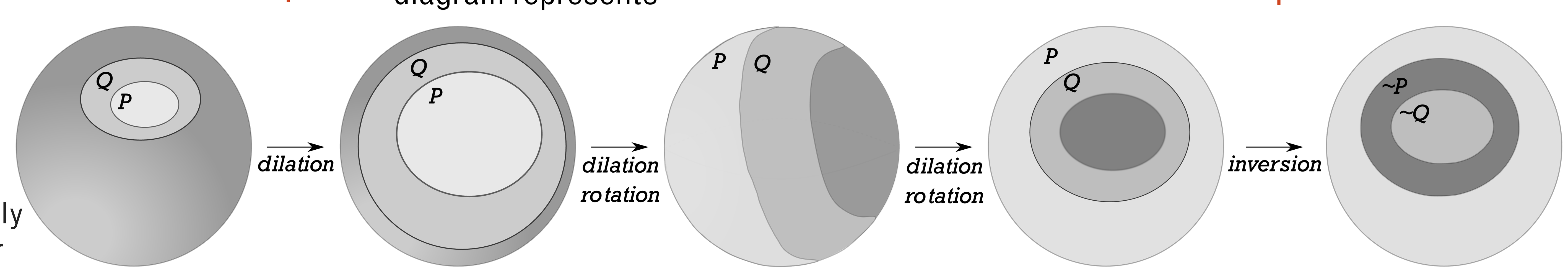


Figure 3: Transforming Euler diagram for $p \to q$ on a sphere to show equivalence to its contrapositive

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References

Proofs Project

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