Students' Enumerations of Tiles within a Rotation-Based Tiling of the Plane: Juxtaposing Units Coordination and Spatial Structuring Perspectives

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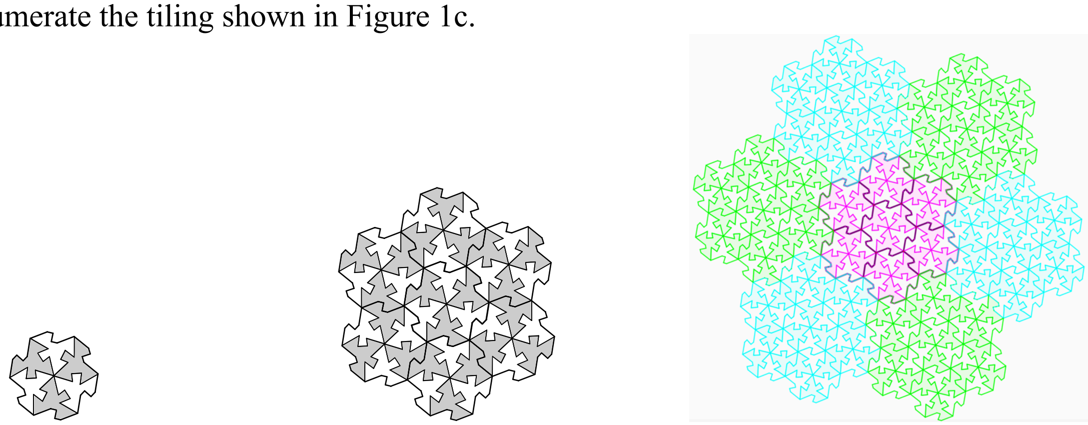
This report contributes novel insights into how undergraduate students think and reason about the number of tiles within a given tiling of the plane. We juxtapose two theoretical perspectives—units coordination and spatial-temporal-enactive (or S-) structuring—to provide an analytic account of how two undergraduate students reasoned about the tiling. In particular, this work suggests potential qualitative differences in how undergraduate students might engage in multiplicative reasoning in an unfamiliar spatial context.*

A tiling (or tessellation) is a collection of spatial structures that have been arranged in such a way that (a) there are no gaps or overlaps between adjacent structures, and (b) if the collection were repeated indefinitely, it would cover the entire plane. This report focuses on the tiling shown in Figure 1c. Few empirical studies have examined how students make sense of and reason about numerical and spatial units within tilings of the plane. Prior research has examined elementary school students' constructions of spatial tiling units in relation to their constructions of numerical units (e.g., Reynolds & Wheatley, 1996; Wheatley & Reynolds, 1996). Other research has focused on tilings from the perspective of mathematical aesthetics (e.g., Eberle, 2014, 2015). However, to date, the research literature has not yet captured how students, particularly those operating at higher levels of units coordination, form spatial and numerical units to enumerate tiles within tilings of the plane. In this report, we contribute initial insights toward this direction. Specifically, using the perspectives of units coordination and S*-structuring, we aimed to understand what multiplicative reasoning undergraduate students might use to enumerate the tiling shown in Figure 1c.



(a) (b) (c)

Figure 1. (a) Six-tile structure (6TS), (b) 42-tile structure (42TS), and (c) 294-tile structure (294TS)

Conceptual Framework

Reasoning about Numerical Units

The theory of units coordination constitutes the first component of our conceptual framework. Grounded in Steffe's work (e.g., Steffe, 1992, 1994), and subsequently developed by

Hackenberg, Norton, and colleagues into a three-stage[1] theory of students' multiplicative concepts (Hackenberg, 2010; Hackenberg & Tillema, 2009; Norton et al., 2016) and related forms of additive reasoning (Ulrich, 2015, 2016), units coordination posits that students gradually develop increasingly sophisticated ways of approaching (i.e., assimilating) additive and multiplicative situations. Stage 1 students can assimilate problem situations using units of one, and they can use units of one to actively build a unit of units (i.e., a level 2 unit) out of perceptual or imagined material. Stage 2 students can assimilate problem situations using both units of one and level 2 units; they can use these unit structures to actively build a unit of units of units (i.e., a level 3 unit) out of perceptual or imagined material. One indicator of Stage 2 reasoning is a tenuous or fleeting awareness of a three-levels-of-units structure. Stage 3 students can assimilate problem situations using three levels of units. That is, they can anticipate, at assimilation, the multiplicative structure of a situation, and they can reliably and flexibly use three levels of units in reasoning.

To illustrate how students at each stage might reason within a multiplicative context, consider the Bars Task (adapted from Norton et al., 2015). This task was presented to the students discussed in this report. Each student was shown a large bar, a medium bar, and a small bar. Placed end-to-end, three medium bars were of the same length as one long bar, and four small bars were of the same length as one medium bar (all bars had equal width). Stage 1 students can characteristically iterate the small bar (using number word utterances or motor-kinesthetic pulses) to determine how many small bars make up one large bar (12). Stage 2 students can iterate the medium bar, with each iteration representing a unit of four units (small bars). Stage 3 students can immediately determine that 12 small bars make up one large bar, with the understanding that the large bar consists of four medium bars, and each medium bar is equivalent to three small bars.

Reasoning about Spatial and Combinatorial Units

To date, little research has examined students' units coordinating structures within spatial/geometric contexts, with some notable exceptions. Wheatley and Reynolds (1996), for instance, found strong parallels between elementary school students' constructions and coordinations of numerical and spatial units. While prior research identified the unitization action as fundamental to the construction of *numerical* concepts (Steffe, 1992; Steffe et al., 1983; von Glasersfeld, 1982), Wheatley and Reynolds also found this action, in addition to mental imagery, to be critical within *spatial* contexts.

Outside of but related to the literature on units coordination, substantial research has aimed at understanding the mental operations and strategies that students use when engaged in tasks for enumerating spatial structures in contexts of geometric measurement (e.g., Barrett et al., 2017; Battista, 2004, 2007; Battista & Clements, 1996; Battista et al., 1998; Clements, 2004; Cullen et al., 2018; Smith & Barrett, 2018). We define the term *spatial unit* to mean a perceived or conceived object in space that is conceptualized as a singular entity (through an act of unitization). How one reasons about spatial units depends on how the individual mentally organizes, or structures, those units. Battista and Clements (1996) defined *spatial structuring* as "the mental act of constructing an organization or form for an object or set of objects" (p. 282). Without a viable spatial structuring to guide their enumerations of spatial units, students may experience difficulty keeping track of spatial units, which can lead to double-counting, or missing, some spatial units. Battista et al. (2018) defined *spatial-numerical linked structuring*

[1] Hackenberg's usage of the term "stage" is consistent with von Glasersfeld and Kelley's (1982) definition.

(SNLS) as a form of numerical reasoning that is linked to one's spatial structuring. From the perspective of units coordination, Zwanch et al. (in press) found that students with the second multiplicative concept (comparable to Stage 2; see Ulrich, 2015) may experience difficulty maintaining an awareness of a three-levels-of-units structure in geometric contexts, such as conceptualizing a strip of paper as a unit, each containing 7 units (7 inches), which each contain 2.5 units (roughly 2.5 cm per inch). Thus, enumerating spatial units depends on one's spatial structuring of the relevant spatial information *and* on one's available units coordinating structures. We hypothesize that the former may depend on the latter—that is, how one spatially organizes units depends on their current additive and multiplicative concepts.

In our research on students' combinatorial reasoning (Antonides & Battista, 2022), we elaborated Battista and Clements' (1996) construct of spatial structuring, defining a new construct called *spatial-temporal-enactive structuring* (or *S*-structuring*). We also defined S*NLS as the analog to SNLS, using S*-structuring in place of spatial structuring. S*-structuring focuses on how students construct, organize, and operate on composite units (level 2 units) and composites of composite units (level 3 units). Antonides and Battista (2022) identified and illustrated two specific forms of S*-structuring. *Intra-composite structuring* entails organizing units into individual composite units (such as combinatorial outcomes)—that is, actively constructing level 2 units out of level 1 units. For instance, within our framework, students engage in intra-composite structuring when they apply a pairing operation (Tillema, 2013) to append objects together to form a combinatorial outcome (e.g., a shirt-pant "outfit," a two-card hand, or a three-cube tower). *Inter-composite structuring* entails organizing composite units into composites of composite units—that is, actively structuring level 3 units out of level 2 units. For instance, students might construct all six 3-object permutations without repetition, organized into three groups with two permutations (i.e., two level 2 composites) in each group.

Methodology

The data discussed in this report come from a larger study (Antonides, 2022) that examined five undergraduate students' reasoning about enumeration. Each student participated in one-on-one clinical interviews and teaching episodes. Pre- and post-assessments took the form of semi-structured interviews (Clement, 2000), and the remaining sessions with students took the form of teaching episodes. During teaching episodes, direct instruction was not the general pedagogical approach; rather, each student was posed with tasks, questions, and scenarios (e.g., digital environments) that we hypothesized, based on our current models of their mathematical understandings, would provoke an accommodation to their current ways of operating. Two tasks are relevant to the current report.

> *Rotational Tiling Task*: Write an expression for the number of tiles in the tiling shown here [students were shown the image in Figure 1c]. One tile is a shape like this [they were shown an example of one tile].
>
> *Extension Task*: If this tiling pattern were to continue, how many tiles would be in the next step of the tiling?

Data were collected remotely, via Zoom, in the Spring 2020 semester. Students used a laptop and an iPad in all video conference calls: the former to capture facial expressions and gestures, and the latter to capture students' inscriptions. Sessions were recorded to capture audio/video data.

This report focuses on the cases of Claire and Kira (pseudonyms). Both students were in their first year, were 18 years of age, and used she/her pronouns. Claire was an elementary education major enrolled in a first-semester content course for future teachers (focusing on number and operation), and Kira was a psychology major enrolled in an intermediate algebra course (a prerequisite for college algebra). All data pertaining to students' solutions to the Bars Task (see the Conceptual Framework) and to the Rotational Tiling Task were transcribed. Analytic memos and codes (Maxwell, 2013) were used to make inferences about the data. Particular attention was given to the types of numerical units that students seemed to construct, and on which they seemed to operate in order to build higher-level unit structures. We further identified moments in our data where students seemed to engage in S*-structuring, and we made inferences about the nature of students' S*-structuring.

Findings

Our analyses of the Bars Task suggested that all students were operating at units coordination Stage 3. Knowing that three medium bars make one large bar, and that four small bars make one medium bar, each student quickly reasoned that 12 small bars would make up one large bar, with the understanding that the large bar could be seen as three groups with four small bars in each group. It should be noted, however, that the Bars Tasks have been validated as a written instrument for assessing the units coordinating structures of middle school students (Norton et al., 2015), but not undergraduate students (including preservice teachers). To date, no such written instrument has been validated for the latter population.

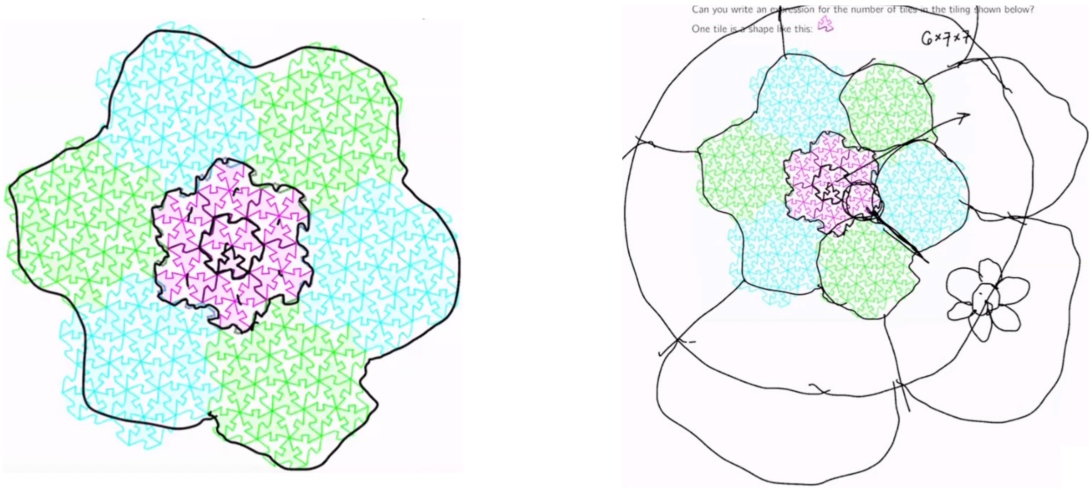
In describing our results and inferences, we use the terms 6TS to refer to a tiling structure consisting of six tiles (e.g., Figure 1a), 42TS to refer to a tiling structure consisting of 42 tiles (e.g., Figure 1b), and 294TS to refer to a tiling structure consisting of 294 tiles (e.g., Figure 1c), all with six-fold rotational symmetry.

The Case of Claire

Excerpt 1: Rotational Tiling Task. The interviewer read the Rotational Tiling Task to Claire, who immediately identified several embedded multiplicative group structures.

C: So, one of these. [She outlined a tile in the center 6TS; see Figure 2a.] The first time I would group those would be in this shape. [She outlined the entire 6TS.] And it looks like there's six of them. And then, the next grouping, I would say, is probably this shape. [She outlined the center 42TS.] And, in this one, it looks like there's one, two, three, four, five, six, seven [6TS structures]. So I'd probably say six times seven. … And then, I would do the same thing, but with this. [She outlined the entire 294TS.] And, um, I would define groups by the different colors. So this one [the center 42TS], in general, looks pretty pink. This one green, blue, and they all appear to be the same shape. And this also has seven [copies of 42TS in 294TS; see Figure 2a].

Claire's final answer to the Rotational Tiling Task was the written expression $6 \times 7 \times 7$.



(a) (b)

Figure 2. (a) Claire's illustration of four levels of units, and (b) her extension of the tiling pattern

Excerpt 2: Extension. The interviewer then read the Extension Task to Claire.

C: So, it would expand outward, I think. … I think there would be another ring, like this. So since this has, like, a *seam*, almost, with each, like, circle here [she drew a line segment separating two exterior 42TS in the original figure; see Figure 2b], I would probably say the same thing. It'd be like that… and another seam on each sort of thing.

Claire drew six large round figures along the exterior of the original tiling figure.

C: And then, I guess within each one, it would be all the levels before it. So, let's see. They would all start with an individual, and then a circle [6TS], and then this [42TS]. And then, like, this part [294TS]. … I guess I could probably copy it.

She then modified her multiplicative expression to $(6 \times 7 \times 7) \times 7$.

Analysis. In Excerpt 1, Claire established a multiplicative relationship between single tiles and one 6TS, constructing a level 2 unit (an instance of intra-composite structuring). She realized one 42TS could be made by iterating 6TS seven times, meaning she constructed 42TS as a level 3 unit. We interpret this as an instance of intra- and inter-composite structuring since she established a new unit as an organization of composite units. She expressed the number of tiles within one 42TS as 6×7. Finally, she established the entire image (one 294TS) as a level 4 unit structure (another instance of intra-/inter-composite structuring). To count the total number of tiles, she operated on her prior multiplicative expression, producing $(6 \times 7) \times 7$.

Claire's robust spatial/multiplicative reasoning becomes especially salient in the Extension Task. In this, she introduced the notion of a "seam" and used this concept to spatially structure units of 42TS within the 294TS. She then imagined, in mental activity, iterating units of 294TS along each seam, producing a new (level 5) unit structure—an instance of intra-/inter-composite structuring. Further, made clear in her verbal explanation and drawing in Figure 2b, she maintained the unit structures nested within each iteration of the 294TS.

The Case of Kira

Excerpt 3: Rotational Tiling Task. Kira approached the Rotational Tiling Task by first focusing on and counting the number of larger groups (42TS) in the figure.

K: So, I'm thinking I could count the number of tiles in each group. And then, um, multiply that by seven. And it would tell you the total number of tiles altogether.

To count the number of tiles in a larger grouping, she focused on the center 42TS. She counted six tiles in one smaller grouping (6TS), and she counted seven 6TS in the central 42TS.

K: So, six times seven, is 42. So, it looks like there's 42 tiles in one group. So… seven times 42, I feel like… would tell the number of tiles… tiles altogether? But just to be sure, I'm going to count the number of tiles in one of them, on the outside, because I don't want to just make that assumption about all of them.

To count the number of tiles in another 42TS, Kira shifted her focus to a 42TS on the right side of the tiling figure. She counted six tiles within one 6TS, and she counted seven 6TS within the grouping. She said $6 \times 7 = 42$, which she said was the same number of tiles that she found before. Feeling more confident that there are 42 tiles in each larger grouping, she concluded there are 42×7 tiles in the entire tiling. Using a calculator, she found the product to be 294.

Excerpt 4: Extension. Kira was then posed with the Extension Task.

K: So, like, would there be six more on the outside?

Int: Six more what?

K: Of the groups I've created, like the one, two—it looks like there's a group of 42. So, would there be six more of those groups on the outside, kind of?

Kira drew a picture, showing six 42TS structures added along the outside of the tiling figure, wedged between pairs of green and blue 42TS structures (see Figure 3).

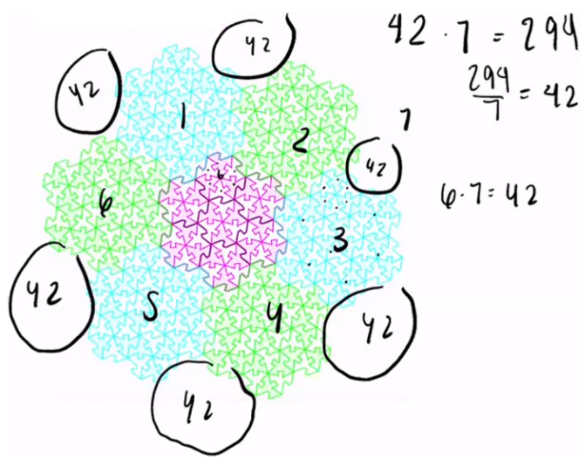


Figure 3. Kira's illustration of how the tiling pattern would continue

K: Now that I'm looking at the image again, I notice that it's like, odd, which is bothering me. So I don't know if there would be another circle somewhere else. But, I'm guessing since it's the same… number? Like the pattern of seven would continue.

Kira seemed to notice that there were seven 42TS in the original tiling, with seven 6TS in one 42TS, and from this she seemed to expect the "pattern of seven would continue." However, unlike Claire who conceptualized the original tiling image as a unit, and who iterated that unit so that seven copies were represented on the screen, Kira anticipated a seventh additional 42TS would need to be drawn, though she was unsure of where to draw it.

K: So I don't know where that one would go when it would continue after that. But, not looking at the image, I would just, um, take my original problem, 42 times 7, and make it 42 times 14? And I feel like that would give the number of tiles in total.

Int: Okay. Where's the 14?

K: Um. So the 14 came from seven plus seven, the number of tiles, if it would be—like this is me not looking at the image. Just, in my head. Since there were seven, I guess it would be, um, another seven. So 14. And there's 42 tiles in each group … 588.

Analysis. In Excerpt 3, Kira conceptualized the entire tiling as a level 3 unit structure. This was indicated by her strategy: count all tiles in one larger grouping (42TS), then multiply by the number of larger groupings (7). In counting the tiles in 42TS, Kira realized it consisted of seven smaller groupings (6TS), and so she multiplied 6×7, and thus she established 42TS as a level 3 unit. Unlike Claire who seemed to immediately anticipate a level 3 unit structure, Kira seemed to establish this unit structure in activity. She planned to operate on this level 3 unit by multiplying it by seven, but she expressed uncertainty. She became more confident only after performing 6×7 to count tiles in another 42TS. Even after she multiplied $42 \times 7 = 294$, finding the number of tiles in the original tiling image, she felt the need to check her answer by applying the inverse operation, $294 \div 7 = 42$, shown in Figure 3.

In Excerpt 4, Kira first extended the tiling pattern by constructing a spatial structuring with six additional 42TS. However—presumably noticing that there were seven 42TS in the original image—Kira anticipated that seven, rather than six, additional 42TS should be added in the extension, though she could not determine where to place the seventh structure. From this, she suggested there would be 42×14 tiles in the extended pattern. Kira, unlike Claire, did not seem to imagine a level 4 or level 5 unit structure. Rather, she created additional level 2 units (42TS) to create a larger level 3 unit.

Discussion and Conclusions

In this report, we set out to examine two undergraduate students' reasoning about the enumeration of tiles within a tiling of the plane using two analytical perspectives: units coordination and S*-structuring. While much work still needs to be done toward elaborating the relationship between these two perspectives (an avenue for future research), we believe this report offers initial steps toward this goal.

Claire established the tiling image in Figure 1c as a level 4 unit, clearly articulating and illustrating the multiple levels of units that she abstracted. She made several acts of intra-composite structuring and linked intra-/inter-composite structuring along the way, establishing 6TS as a level 2 unit, 42TS as a level 3 unit, and ultimately 294TS as a level 4 unit. Note that each level of unit is described as such because Claire could think about the four levels of multiplicative relationships that she built up to establish 294TS. She extended the tiling pattern by taking the 294TS as an object that she could iterate, guided by her spatial structuring of the original figure that was mediated by her notion of a "seam." Overall, Claire demonstrated S*NLS reasoning of a student who could establish, interrelate, and iterate four levels of units to construct a level 5 unit structure.

Kira enumerated tiles in the original tiling by establishing it as a level 3 unit—a unit of seven units, each containing 42 units. She separately counted each 42TS as a unit of six units, but she did not maintain this relationship, and it was not used in her enumeration of tiles in the tiling. In the Extension Task, Kira's S*NLS was distinguished from Claire's since her S*-structuring of the extended tiling was a level 3 unit—a unit of 13 (or 14) level 2 units—rather than an iterated level 4 unit.

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