# Contextuality in Reasoning about the Exponential Function: A Microgenetic Learning Analysis

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*This paper explores how two undergraduate students come to perceive exponential structure in a series of word problems. This study investigates whether features of task situations can create opportunities for supporting students in recognizing situations as appropriate for using exponential models. The analysis is guided by coordination classes theory (diSessa & Sherin, 1998) and Transfer-in-Pieces perspective (Wagner, 2006; 2010), which emphasizes the role of multiple instantiations of a concept across multiple contexts as learners construct their understanding. The results suggest that contexts that embody the notion of recursive dependency may help students activate and use appropriate knowledge resources related to exponential models.*

*Keywords*: Exponential models, Context-dependence, Knowledge in Pieces

## Introduction and Purpose of the Study

There is an essential need to explore students' understanding of different topics at the college level, especially those topics that students find difficult. Exponential functions are a difficult, yet essential, mathematical concept that plays an important role in the study of advanced mathematics (Ellis et al. 2016; Weber 2002). Several researchers illuminated the importance of deep understanding of exponential function as an important topic in the mathematics education field (Brendefur, Bunning, Secada, 2014). Weber (2002) focused on students' initial understandings of the exponential function. The main result from Weber's (2002) work was that although all students could compute exponents in simple questions, only few students could reason about the process of exponentiation. The result stimulates some essential questions regarding understanding the exponential function: What features of contextual tasks do students notice and how does this impact their reasoning patterns? In particular, what features of tasks meant to cue the use of exponential models are salient to students? How do students' reasoning patterns shift as they engage with a sequence of tasks designed to help them link their prior experience with linear modeling to exponential modeling?

## Theoretical Framework

This work requires a perspective that brings into focus on the role of contextuality of a task in students' thinking and reasoning. To this end, I turned to Wagner (2006; 2010) who addressed the role of knowledge in transfer processes. Wagner (2006) demonstrated in "transfer-in-pieces" that prioritizing context sensitivity of knowledge supports transfer. Wagner (2006) framed his analysis using the construct of *coordination classes* which is rooted in diSessa's (1993) Knowledge-in-Pieces (*KiP*) heuristic epistemological framework. Coordination classes give a model for conceptual understanding that involves how individuals extract concept relevant information from a context. These constructs can help to explain how learners initially interact with mathematics content in a wide range of contextual tasks. In describing the knowledge that emerged in this study, the coordination class theory construct of *concept projection* proved to be useful. A *concept projection* is, “the particular set of strategies and cognitive operations that are used by an individual in applying his or her concept (coordination class) in a particular situation…” (diSessa & Wagner, 2005, p. 128).

Wagner (2010) provided an extension of transfer- in-pieces in which the alignment with the Piagetian frame is explicitly pointed out to create a framework for transfer either in similar situations or accommodation of a new concept. “Concept projections are thus collections of assimilatory and interpretive knowledge elements associated with some concept, and the construction of new concept projections reflects a process of conceptual accommodation to new contextual situations” (Wagner, 2010, p.451). Building on Wagner’s (2006, 2010) perspective, I explored how learners can build new knowledge elements —exponential understanding—which is the result of passing through multiple disequilibrium and accommodating particular context. In this work, another goal is to investigate the affordance of context for such accommodations. In other words, I seek to explore from the novice’s perspective what is the feature of contexts that exponential understanding can be accommodated to? The affordances of a context are a fundamental aspect of investigation of coordination classes and transfer. “Transfer requires that the structure perceived in a new situation be matched or closely matched to the structure perceived in a previously encountered situation” (Wagner, 2006, p. 5). A major difficulty is the explanation of contextual situation which can be matched to the ultimate, desired abstraction (Wagner, 2006).

In this work, I investigate the affordances of *recursive dependency* implicit in very common exponential contexts for instigating productive conceptual conflict. I define recursive dependency as a feature of a context task in which reasoning about each step depends on the previous step. As an example, the molding process of the population of a city, increasing by a particular rate every year, needs learners to realize that the number of people of the city in each year *depend on the previous* year, rather than the initial year. I argue that the *recursive dependency* is the heart of, specifically, identification of exponential function as it is shown in two undergraduate students’ reasoning in this study.

## Data Collection

Data for this study were taken from semi-structured task-based interviews of four undergraduate students who enrolled in a developmental math course—Algebra II—at a large, public, Midwestern university. The interviews were conducted in an online format consisting of 60-100 minutes of problem solving. The interview involved screen sharing a specially designed set of slides I created through Desmos Activity Builder (DAB). Participants agreed to meet with me for 60-90 minutes one-on-one in an online private session. They worked in DAB and their thoughts as he described as they worked on a contextual problem involving exponential function was audio and video recorded. Screenshots of all written work in DAB were saved and some are represented in appendix A. All 60-100 minutes of the interviews were transcribed for detailed analysis.

The main interest to this research was moments in which participants’ patterns of reasoning appeared to be in transition or if it appeared that they came to notice a conceptual conflict. Among four participants there was one student, Leia, who understood the context qualitatively in City A problem and calculated the population after 10 days and 20 days, however, introduced a linear model and stuck with a linear model. Another student, Nick, who already knew exponentiation in different contexts and models exponentially. Two students, Tom, and Andrew, who underwent conceptual change and came to discern exponential function in different contexts. This study focuses on how Tom and Andrew underwent some conceptual change to understand exponential function.

Among the problems presented to students were sets of common contexts for exponential situations, like population growth and the amount of a drug remaining in a patient's body. However, there is a special effort for raising the possibility of more engagement from each student. For example, city A and City B problems, presented in Table 1, is an explanation of the population of people who are infected by Covid 19. This study happened during the peak of the pandemic, when checking the number of Covid 19 outbreaks was a concern. the set of problems that students attempted to solve in the 60-100 minutes interview. I sometimes asked probing questions, however, I avoided taking an evaluative role and emphasized my appreciation for valuable sharing of their thinking with me. An analysis of explanatory language that Tom and Andrew used during the interviews enabled me to recognize the importance of well-matched recursive dependency in the exponential context to generate productive disequilibrium and support transfer.

*Table 1. Exponential problems that participants worked on in the 60-100 minutes interviews.*

| Exponential problems |
|---|
| <u>Baseline problem</u><br>What does the function $f(x) = b^x$ mean to you? What do you think of when you see this function? |
| <u>People Diagnosed with Covid-19 in City A</u><br>In City A the number of people who have Covid-19 increases by 11% every 10 days. The initial population of people who are diagnosed with Covid-19 in City A is 1500. Create an equation that models this situation. |
| <u>People Diagnosed with Covid-19 in City B</u><br>In City B the number of people who have Covid-19 decreases by 11% every 10 days. The initial population of people who are diagnosed with Covid-19 in City B is 1500. Create an equation that models this situation. |
| <u>Drug Remaining in the Body</u><br>Imagine that a patient is taking 100 milligrams of a drug. Suppose only 27% of the drug remains in the patient's body after an hour. How long does it take before only 10% remains? |

## Analysis

Microgenetic Learning Analysis (*MLA*) is an analytic methodology for progressing from observations in video data of moment-by-moment reasoning processes to systematically describing the nature and form of knowledge growth and change. (Parnafes & diSessa, 2013). My study also can be summarized as MLA of two undergraduate students' (Tom and Andrew) reasoning, coming to understand and discern the appropriateness of contexts for modeling exponentially. In this work, the MLA process includes transcribing the episodes, enriching them with details about what was written and pointed to in Desmos by participants, and then summarizing the work on each task and the discussion around it. Describing learners' first reaction for a modeling system and how they both modeled linearly; then playing genuinely just with context helped them to go through an assimilatory activation and, ultimately, accommodate a new reality which was concept projection of exponential function. I defined this process of, first, reacting proportionally to, eventually, accommodating exponentiation as the process of "changing". As I observed, both learners could experience this "changing" process either in City A problem (Tom's experience) or in City B problem (Andrew's experience) with assistance of productive conceptual conflicts in which a well-imbedded recursive dependency appeared as a highly influential factor for change.

**Analysis of Tom's reasoning**

**City A problem: Facing conceptual conflicts (disequilibrium).** Coming to solve the problem of city A is the heart of this study, in which we can see the prominent pattern of logical readout of the situation, coordinating the context and his pattern of reasoning, experiencing a productive disequilibrium, accommodating new understanding of the situation, and finally, generalizing the problem to make an abstract model. To me, the key observation across the session was the moment that Tom in the middle of making sense of the structure leaves all his findings (including the population of Covid 19 after 20 days: getting 11% of 1665 and adding to 1665, not 1500, which had the result of 1848) and faces a huge *disequilibrium* with satisfying a systematic and already established proportional perspective.

Interviewer: How did you find that number?
Tom: I divided 1848 by 1500 and found 1.321 which does not make any sense
Interviewer: what about first 10 days
Tom: That works really well. 1665/1500 is 1.11 which makes a lot of sense. I thought 1848.15/1500 should be 1.22 because it increases 11% every 10 days

The *recursive dependency*, well-imbedded in the context task, appeared to help Tom to prefer working with his understanding of context rather than sticking with his proportional expectation. He eventually could coordinate his perception of the situation with his strategy for calculating an increasing Covid19 population in each step and to resolve an apparent contradiction in his thinking.

Tom: Ok! So, oh! Wow! That's a lot! Hah! Ok! So, in the first step we are taking 11% of 1500 and then plus 1500 which is $1.11 \times 1500$ and then in the second step we do $1.11^2 \times 1500$ and in the third step it should be $1.11^2 \times 1500$ right? Is that it?
Tom: SO! f (x) $= 1.11^x \times 1500$ would be: Right? Cool.
Interviewer: Do you want to calculate what you find in the second step? I guess it would be interesting for you
Tom: Sure! It is 1.321! What? What is that? ***Is it magic?***

**City B Problem: Testing Near Transfer.** In Tom's work on the City B problem, he showed transfer more explicitly in a similar context. I asked him to calculate the population of people who have covid 19 after 10 days. While he was calculating it seemed that he already had the exponential model in his mind.

Tom: Ok! I found it! If I take this (1.11 from the previous model), because it is one point eleven! If I want to decrease (Looking ups and down)! hundred mines 11… like 89 percent? If

I do 0.89 (More confidently and surely say the last words) that gives me the model here, the 11 percent decrease (laughing happily!)

Interviewer: Do you want to write the model?

Tom: Yes, sure! I think it should be f (x) $= 0.89^{x} \times 1500$ , right?

**Remaining Drug Problem: Fidelity to Proportional Modeling when Context Lacking Recursive Dependency.** After 73 minutes of productive challenge to understand exponential function, Tom confidently initiates reasoning proportionally. In the end, he modeled this problem with a linear and solved a linear equation to figure out how long it would take for 10% of the drug to remain. Figure 3 in appendix is a screenshot of Tom's work when he was sharing his problem solving on the WebEx. I tried to have a conversation to see why the first reaction to this question involves linear thinking. I asked him why he thinks that a linear model can work in this problem, and he answered that "I am not an expert, but our body must work linearly. It cannot work first slowly and then [become] hardworking!" At this moment in Tom's interview, I could have asked "why?" (Why can the body not work with different rates of the drug?) However, it did come to my mind as I was experiencing the first observation of how learning is highly contextual. In the interviews with Andrew which were conducted after Tom's, I was able to get more insight.

**Analysis of Andrew's reasoning**

**City A Problem: Appropriate Use of Exponential Model.** Andrew found a strategy to calculate the population of Covid19 immediately, however, just like Tom, after seeing some steps in the exponential model, he checked himself with a proportional model. Facing a contradiction between understanding of the problem and the linear model, I asked him what his decision is.

Interviewer: If I want you to calculate after 30 days, how do you go, are you plugging 3 in your linear model or you are calculating from the first strategy and why?

Andrew: I would probably do the first one, because the way that I understand the increases, it is not that taking the old percentage and taking 330, it is taking the ***11% from the new number.***

I observed that a well-matched recursive dependency emerged to help Andrew to decide his strategy and confidently leave proportional thinking. Like Tom, Andrew eventually modeled exponentially this problem.

**City B Problem: Facing a Multi-dimensional Disequilibrium.** Andrew recognized what decreased means as he could see $0.89 \times 1500$ for the first set of 10 days and $0.89 \times 0.89 \times 1500$ for the second set of 10 days. I thought he was ready to make the model, just easily in a similar context; however, he was still in disrepair with the linear model.

Andrew: I think I could! it would be 1500 times (X-0.11)? so for the second step it would be 1500 times 2-0.11, but no! that would be an increase! No?! It cannot be correct, because it is two thousand and something which is not even a decrease.

Interviewer: Ok let's talk about your strategy, not the model… so far you found 0.89 … how about after 20 days?

There was an implicit point here in Andrew's reaction, in this episode. Although the similarity of City A and City B problem should have resulted in the transfer for abstraction more immediately, there was a tacit avoidance in Andrew's reaction in which, I believe, he was hesitating to connect a decreasing process to an exponential model. I think there was an intuition that if a number comes to the power of something, it should be an increasing process. Andrew was facing multiple conflicts simultaneously; that is why I made up the "multi-dimensional disequilibrium" for his reasoning pattern in the City B problem. Eventually, he could see the model after working on the population after 30 days.

**Remaining drug problem: Fidelity to Proportional Modeling when Context Lacking Recursive Dependency.** Like Tom, after 63 minutes of productive struggle for understanding the exponential model, Andrew returned to proportional thinking very confidently. He made the proportion for finding the slope and then a linear model. This time I had that crucial "why" to ask, moreover, involving more of learner's voice in the study:

Andrew: I was assuming a consistent rate
Interviewer: *Why* do you think it has a consistent rate?
Andrew: Based on kind of information that I get
Interviewer: So, imagine in the question we have this assumption that the drug is working exponentially in the body, how would your answer change?
Andrew: Exponentially cannot work here, because it just says this much left, or this much remains after the one hour.
Interviewer: What do you expect from a problem for model exponentially
Andrew: I think, special kind of increase or decrease

It turns out that "special kind of increase and decrease" is related to dependency to the previous step that I defined as recursive dependency.

## Discussion

The analysis of Tom and Andrew revealed prominent patterns of logical readouts in the situation, coordinating context to reasoning, experiencing productive disequilibrium, accommodating new understanding of the situation, and finally, generalizing problems to make abstract models. During their reasoning about the contextual problems, both learners attempted to create a coordination between the context he perceived and a strategic pattern of thinking to solve the problem.

In the city A problem, there was a moment, in the first interview, when Tom could have made a generalization, but instead he faced a huge **conceptual conflict (disequilibrium)** with a previous normative and settled way of thinking:  proportional reasoning. Andrew, however, experienced this productive disequilibrium majorly in the city B problem. Eventually, playing genuinely just with context helped Andrew to face a multiple disequilibrium to go through an **assimilatory activation** and, ultimately, accommodate a new reality which was **concept projection of exponential function.** Here, the strength of the context can be illuminated as the most important key, which can shadow the importance of similarity.

I conjecture based on the results of both interviews that if the context could have provided enough support for exponential thinking, perhaps students also could have moved through a possible conceptual conflict and accommodate the desired model to the context.

However, when the context was weak in establishing a recursive dependency, learners strongly coordinated the proportional reasoning and modeled the situation linearly.

To elaborate, the learners' mathematical behavior demonstrated a natural product of amplified understanding in the analysis, which is the core of transfer. Understanding a concept is, indeed, inseparable from developing logical readout strategies and **coordination of knowledge resources**. The analysis of Tom's and Andrew's exponential understanding is thus consistent with Wagner's (2006; 2010) Transfer-in-Pieces, where readout strategies and coordination of knowledge resources were matched with affordances of the contexts. After analyzing the interviews, I argue that the Remaining-Drug problem failed to afford recursive thinking. That is, the focus of the situation on the quantity "remaining," "staying," or even "half-life" could not create understanding of the amount of the remaining drug in each hour depending on the amount of the remaining drug in the previous hour. However, in contrast, the growing population of City A demonstrated a well-imbedded recursive dependency and afforded a dependence of the increased population to the previous increased population.

## Implications

Such revelations about the interplay between contexts and the knowledge resources participants activate and use, is an important line of work that has practical relevance with respect to teaching and learning. Such analyses underscore the need for more epistemological investigations across disciplinary topics that can lend insight into the organization of learners' perceptions and inferences.

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