The Need for Conditional Reasoning Tasks with Mathematical Content

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This theoretical report argues that conditional inference is central to undergraduate mathematics, and that there is potential for using and adapting conditional inference tasks from cognitive psychology to study ways in which reasoning develops (or does not develop) with mathematical expertise. Specifically, developing mathematical expertise might improve conditional reasoning in mathematics only, or it might improve conditional reasoning in a way that transfers to abstract and/or everyday contexts; alternatively, it might develop conceptual understanding, so that people become better at conceptually valid mathematical inferences but more vulnerable to invalid responses typically seen in tasks with everyday content. This report argues that, to test and distinguish these possible developmental mechanisms, we need new conditional inference tasks with mathematical content. It also describes the first stage of a project designed to develop (and later use) such tasks, initial outcomes of which will be reported at the conference.

Keywords: logic, reasoning, conditional, if-then

Introduction

Conditional reasoning is central to undergraduate mathematics, especially in upper-level courses. Students learning to understand and construct proofs need to work in mathematically valid ways with conditional (if-then) statements. They need, for instance, to make standard interpretations in relation to implicit quantification. Mathematically experienced people tend to infer universal quantification, reading "If 3 divides x, then 6 divides x" as (something like) the proposition "For all $x \in Z$, if 3 divides x, then 6 divides x" and thus interpreting the statement as false (Hub & Dawkins, 2018; Solow, 2005). Less mathematically experienced people might not infer universal quantification, and reasonably claim that the truth value of "If 3 divides x, then 6 divides x" is undetermined because it depends on the value of x (Durand-Guerrier, 2003).

Importantly for this report, students also need to recognise that some inferences from conditional statements are valid and some are not. Four standard inference types – *modus ponens*, *denial of the antecedent*, *affirmation of the consequent*, and *modus tollens* – are illustrated below.

Modus Ponens (MP)
If x is less than 2, then x is less than 5;
x is less than 2;
Therefore, x is less than 5.

Denial of the antecedent (DA)
If x is less than 2, then x is less than 5;
x is not less than 2;
Therefore, x is not less than 5.

Affirmation of the consequent (AC)
If x is less than 2, then x is less than 5;
x is less than 5;
Therefore, x is less than 2.

Modus Tollens (MT)
If x is less than 2, then x is less than 5;
x is not less than 5;
Therefore, x is not less than 2.

With universal quantification assumed, MP and MT inferences are valid under the mathematically normative material interpretation of the conditional, in which a conditional statement is false if and only if its antecedent is true and its consequent is false (Velleman, 2006). DA and AC inferences are not valid, but endorsing them is natural if the conditional "if A then

B" is interpreted as the biconditional "A if and only if B" (Blockowiak, Castelain, Rodriguez-Villagra & Musolino, 2022).

In cognitive psychology, conditional inference has been studied extensively via tasks that systematically vary content across the four inference types (Oaksford & Chater, 2020). Some research uses everyday content with statements like "If Jenny turns on the air conditioner, then she feels cool" (De Neys, Schaeken & d'Ydewalle, 2005). Other research uses abstract content with imaginary letter-number pairs and statements like "If the letter is A then the number is 1" (Evans, Clibbens & Rood, 1995). Participants are typically asked whether or not they endorse inferences of the four types. With variations depending on factors such as believability of the conditional statements (Evans, Handley & Bacon, 2009) and availability of counterexamples (Cummins, 1995; De Neys, Schaeken & d'Ydewalle, 2003), people often endorse inferences that are not normatively valid.

Departures from normatively valid interpretations need not be problematic in everyday life, where we tolerate ambiguity and rely on context to infer intended meanings (Evans & Over, 2004). Indeed, reasoning research is much occupied with developing models that describe human reasoning more accurately than does normatively valid logic (Oaksford & Chater, 2020). However, departures *are* problematic in mathematics, which makes strict distinctions between inferences available from true conditional statements like "If x is less than 2, then x is less than 5", and those available from true biconditional statements like "x is even if and only if $3x$ is even". Failure to recognise invalid inferences can lead to making or accepting invalid mathematical arguments (Inglis & Alcock, 2012; Selden & Selden, 2003), so mathematicians care greatly about valid conditional reasoning.

Wider society cares about valid reasoning too, and mathematical education is valued partly because there is widespread belief that it develops logical reasoning skill in general (Inglis & Attridge, 2016). This is known as the *theory of formal discipline* and it is certainly plausible. Mathematics has clean conceptual boundaries (Vinner, 1991): a number either is or is not less than 2. So logical relationships in mathematics are clear-cut in a way that everyday ones are not. Consequently, studying mathematics – especially at advanced levels – might help people to become sensitised to logical structures and to transfer this understanding to general contexts. Empirical evidence on this process is, however, minimal and mixed, as described below.

Conditional Reasoning and Mathematics

Research indicates that, as the theory of formal discipline predicts, post-compulsory mathematical study does develop conditional reasoning toward normatively valid interpretations. However, it does so unevenly and imperfectly. Research in the UK (Attridge & Inglis, 2013) and in Cyprus (Attridge, Doritou & Inglis, 2015) found that among 16-18 year-old students, those studying mathematics intensively became better able to reject invalid DA and AC inferences in abstract conditional inference tasks. However, they still endorsed just under half of DA inferences and more than half of AC inferences (Attridge & Inglis, 2013). These students also became slightly more likely to reject normatively valid MT inferences (Inglis & Attridge, 2016), a finding in line with work showing that individuals with higher general intelligence scores and higher overall conditional inference scores tend, if anything, to endorse fewer MT inferences than those with lower scores (Inglis & Simpson, 2009; Newstead, Handley, Harley, Wright & Farrelly, 2004). This means that mathematical study seems to shift students away from naïve interpretations of conditionals but not toward a material interpretation; the shift is better understood as toward a *defective* interpretation in which a conditional statement is viewed as irrelevant when its antecedent is false (Attridge & Inglis, 2013).

Indeed, although individuals might tend to endorse conditional inferences in line with particular interpretations, this does not mean that they apply those interpretations consistently (Evans, Handley & Bacon, 2009). Reasons relevant to pedagogical design are evident in detailed qualitative research in undergraduate mathematics education. For instance, it would be a mistake to think that students considering conditionals with mathematical content are necessarily thinking about logic. Their reasoning might instead be built on content-specific and pragmatic factors (Dawkins & Cook, 2017), so that they benefit from experience designed to focus their attention on assigning truth values consistently across contents (Dawkins & Norton, 2022; Dawkins & Roh, 2022; Hub & Dawkins, 2018). Similarly, it would be a mistake to think that students understand the connections within and between conditional statements and the truth tables often used in teaching. Instead, they might treat the components of both statements and tables as separate entities, which prevents them from effectively reasoning about negation and about conditions under which statements would be false (Hawthorne & Rasmussen, 2015)

We see the impact of imperfect conditional reasoning in undergraduate students' evaluations of mathematical arguments: secondary students and undergraduates often fail to distinguish arguments for the converse of a conditional statement from those for the statement itself (Dawkins & Roh, 2022; Hoyles & Küchemann, 2002; Selden & Selden, 2003). Professional mathematicians do not make that mistake: they reliably identify and reject arguments that prove a converse (Inglis & Alcock, 2012). However, their conditional reasoning remains intriguingly imperfect. They do not, for instance, perform as well as might be expected on the Wason selection task (Wason, 1968). This task involves cards with a number on one side and a letter on the other: participants are shown cards labelled D, K, 3 and 7 and asked which should be turned over to determine whether the rule "every card that has a D on one side has a 3 on the other" is violated. Inglis and Simpson (2004) found that mathematicians largely avoided the common error of selecting D and 3, but fewer than half gave the normatively valid answer D and 7. This task involves evaluating the truth of a conditional rather than making inferences from a conditional that is assumed to be true, but the result is in line with the findings on mathematics students: mathematicians neglect the case corresponding to MT (if not-3 then not-D).

Unsurprisingly, given these findings, there is no standard way of teaching conditional inference. Mathematicians explicitly teach logical reasoning in textbooks (e.g., Houston, 2009; Velleman, 2006) and in introduction-to-proof courses (Davis & Zazkis, 2019), but there is debate on how to do this. Some, for instance, believe that teaching should leverage everyday reasoning and previous mathematical knowledge, others that everyday content muddies the waters and that mathematical education should focus on abstract logical structures (Alcock, 2010; Epp, 2003; Hawthorne & Rasmussen, 2015). Books designed to support students at the transition to proof offer short explanations of truth values in relation to conditional statements, perhaps emphasising the difference between a conditional statement and its converse and explaining why we consider a conditional to be true when its antecedent is false (e.g., Houston, 2009; Solow, 2005; Velleman, 2006). But they do not usually explain any research on everyday reasoning or acknowledge that professional mathematicians apparently need not be perfect logicians. If we wish to support students in developing mathematical expertise, there is pragmatic reason to investigate what that development might look like and where it ends up.

Possible Mechanisms and the Theory of Formal Discipline

Of theoretical interest is that although the theory of formal discipline seems to be partially valid, the mechanism by which it operates remains largely unexamined. The mechanism suggested in the Introduction requires that studying mathematics sensitises people to logical structures and that this understanding transfers to other contexts. This appears not to be fully correct: Inglis and Attridge (2016) suggested that mathematics students' lower endorsement of all three of DA, AC and MT inferences might result from their greater experience of error checking, which leads them to be more productively skeptical about all conditional inferences (see also Inglis & Simpson, 2004). But the larger point stands: if mathematical expertise makes people more productively skeptical, then those with more expertise should perform in similar ways across tasks with all types of content. However, mathematical study could promote different reasoning in mathematics without promoting transfer. If so, people with more mathematical expertise should perform more normatively on mathematical tasks but not on tasks with everyday content. In that case, abstract tasks make an interesting and potentially intermediate possibility: the evidence so far has used exactly these tasks, showing that mathematics students become better at rejecting DA and AC inferences (Attridge & Inglis, 2013). Performance with abstract content might represent the best that we can expect in terms of valid reasoning, or performance might be even better with meaningful mathematical content.

Alternatively, mathematical expertise might be less about sensitivity to logical structures or general skepticism, and more about deeper conceptual understanding: greater familiarity with mathematical examples, definitions and theorems might enable people to "understand the terrain" (cf. Goldenberg & Mason, 2008; Michener, 1978) and avoid errors by thinking about mathematical situations as they would about "real" situations. If so, people with more expertise might exhibit patterns of mathematical reasoning that reflect those seen for everyday content. For instance, conditional inference is known to be affected by believability. Individuals are less likely to endorse inferences of all four types when a conditional is less believable (Evans, Handley & Bacon, 2009). Similarly, conditional inference is affected by counterexample availability. For a causal conditional "If *p* then *q*", people are less likely to endorse DA and AC inferences if there are many *alternatives* that might cause *q* in the absence of *p*; they are less likely to endorse MP and MT if there are many *disablers* that might intervene to prevent *p* from causing *q* (De Neys, Schaeken & d'Ydewalle, 2005). People can usually think of several alternatives and disablers for "If the brake was depressed, then the car slowed down", for instance (Cummins, 1995). If greater expertise makes mathematics more "real", then it should be possible to manipulate believability and counterexample availability in mathematical content so as to induce everyday-like response patterns. This might support an account in which mathematicians do not really need formal discipline because expertise is primarily a matter of content knowledge.

These possible mechanisms for the theory of formal discipline – or for its relative unimportance – are not currently distinguishable because research has not examined conditional reasoning with mathematical content. Cognitive psychologists have almost exclusively used tasks with abstract or everyday content, even when studying mathematics students (Morsanyi, McCormack & O'Mahony, 2018) or investigating the theory of formal discipline (Attridge & Inglis, 2013). Research in mathematics education has been small-scale (Hub & Dawkins, 2018) or not systematically related to all four inferences (Hoyles & Küchemann, 2002). To understand how mathematical expertise develops conditional reasoning, and to relate that development to research in cognitive psychology, we need tasks with mathematical content.

From Theory to Empirical Research

The talk for this report will present the theoretical argument given above and describe the development of two new conditional reasoning tasks with mathematical content. (The larger project of which this work forms part will use these alongside tasks with abstract and everyday content to study how conditional reasoning develops with mathematical expertise.) The new tasks will both comprise conditional inference items of the form shown in the Introduction. One (MCB) will have basic mathematical content familiar to all educated adults; inferences will likely be from conditional statements such as "If x is less than 2, then x is less than 5". The other (MCA) will have advanced content familiar only to people with upper-level undergraduate mathematical experience; inferences will likely be from conditional statements such as "If $f: R \to R$ is differentiable at a then f is continuous at a". Both tasks will be structured using a standard *negations paradigm*, systematically varying negated elements across antecedents and consequents of conditional statements (see Attridge & Inglis, 2013; Evans, Handley, Nielens & Over, 2007).

Specific content for the MCB and MCA will be developed in Fall 2022. Mathematicians experienced in teaching proof-based courses will learn about factors known to affect everyday and abstract conditional inference; they will take these into account when generating potential items. This will not necessarily be straightforward for mathematical content. For believability, there are obvious possible manipulations: switching a true conditional like "If x is less than 2, then x is less than 5" for its false converse "If x is less than 5, then x is less than 2" might interfere with performance in predictable ways; switching it for a conditional with no obvious truth value, like "If x is less than 2, then y is less than 5" might render it more like an abstract task. For counterexample availability, the picture is more complex. It might be straightforward to affect DA and AC inferences: alternatives to "If x is less than 2, then x is less than 500" should be more accessible than alternatives to "If x is less than 2, then x is less than 5", for instance (see Hamami, Mumma & Amalric, 2021, for related results in geometric contexts). But the absolute nature of truth in mathematics makes it difficult to manipulate disablers that might affect MP and MT inferences: nothing could interfere to prevent x being less than 2 from "causing" x to be less than 5, and there is no readily inferable cause in "If x is less than 2, then y is less than 5". It will be interesting to see whether and how mathematicians attempt to address these issues, as well as whether and how they deal with implicit quantification.

With potential items in place, the mathematicians will collaboratively narrow down the list with the aim of constructing MCB and MCA tasks that mimic the systematic variation in tasks with everyday and/or abstract content. This will be accomplished using a Delphi-like process in which the mathematicians review, revise and prioritise one another's suggestions; individual interviews will capture the reasoning behind their suggestions, revisions and prioritisations. Later, the items will be tested with undergraduate participants to establish the extent of their comparability with standard abstract/everyday tasks. The talk for this report will present the outcomes of the Delphi process and the interviews: it will exhibit the items developed and present a qualitative analysis of the mathematicians' input in light of the theory outlined in this report.

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