What is Instantaneous Rate of Change?

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The purpose of this study is to examine students' meanings for the derivative at a given input value. While students may verbally state that the derivative represents instantaneous rate of change, that does not imply that they have a coherent meaning for the difference quotient as an average rate of change. This study explores students' responses to a typical calculus 1 problem that requires the use of derivatives to determine a linear approximation. This study analyzes student responses about instantaneous rate of change and the differences in meanings that students may hold about it.

Keywords: Derivative, Instantaneous Rate of Change, Calculus, Student Meanings

The concept of derivative is foundational to the learning of higher topics in the STEM fields. However, researchers report that the learning of derivative is difficult (Tall, 1981; Park, 2013; Zandieh, 2000; Oehrtman, 2002; Monk, 1994; Ubuz, 2007; Yu, 2020). Therefore research on how students reason about the idea of derivative will be useful for informing efforts to improve the teaching of derivative. The seemingly paradoxical issue that students have to resolve is interpreting the derivative at a given input value ($f'(3) = 6$) as regarding a single instance (at the input of 7), yet derivatives are about change, so how can you talk about change if there is only one instance involved? Perhaps due to this issue, students hold multiple disconnected meanings about derivative (Zandieh & Knapp, 2006) as a coping mechanism to get through their calculus courses. Since derivatives represent something that we call "instantaneous rate of change", then students' meanings for rate of change are pertinent to their understanding of derivative. Yet, researchers (Byerley et al., 2012; Simon & Blume, 1994; Castillo-Garsow, 2010) have shown that students have differing ideas about what a rate of change is. Due to these issues related to learning the idea of derivative and rate of change, the purpose of this study is to explore students' meanings about the idea of derivative as instantaneous rate of change. The research question this study examines is:

How do students interpret the derivative in a context as instantaneous rate of change?

Literature Review

The literature on derivatives has been extensive thus far, ranging from the limiting process (Oehrtman, 2002; Roh, 2008, Ferrini-Mundy & Graham, 1991; Weber et al., 2012), conceptions of rates of change (Byerley & Thompson, 2017; Hackworth, 1995; Thompson, 1994, Confrey & Smith, 1994), and student understandings of the derivative as a function (Park, 2013; Baker et al., 2000; Aspinwall et al., 1997; Asiala et al., 1997; Borgi et al., 2018; Zandieh, 2000).

Derivatives are about quantities changing; thus, how students reason about quantities changing is central to how students construct a meaning for instantaneous rate of change. Castillo-Garsow (2010,2012) investigated how students may be reasoning about change and identified two distinct student images of change, *chunky* and *smooth*. A student engaging in chunky reasoning imagines the value of a quantity changing discretely from one point to another but with no imagery of motion between those two points. A student engaging in smooth reasoning imagines a change in a quantity as a change in progress by conceptualizing a quantity as taking on values as time flows continuously and smoothly. These different ways of imagining changes in quantities would likely influence how a student interprets a rate of change.

According to Zandieh (2000), there are four representations of derivatives that students tend to recall: graphically as the slope of the tangent line, verbally as instantaneous rate of change, physically as speed, and symbolically via the limit definition of derivative. Zandieh's derivative framework has been central in calculus education research (e.g., Petersen et al., 2014; Roundy et al., 2015; Zandieh & Knapp, 2006) and has led other researchers to hypothesize the desired understandings we hope students will hold (Jones & Watson, 2017). While Zandieh's (2000) framework is a useful tool in comparing students' meanings with a desired meaning, one gap in the literature is how students conceptualize the value of an instantaneous rate of change. In order to make changes in the standard Calculus curriculum, we need to investigate the meanings that students are constructing about derivative and analyze how and why students build these meanings. This paper explores how some students explain their meaning for instantaneous rate of change and characterizes the ways of thinking that these students employed.

While a student may verbally explain a derivative as a rate of change, research has shown that students have various understandings about what a rate of change is. Byerley et al. (2012) investigated calculus students' understanding of division and found that many students used additive reasoning when interpreting a rate's value. These students considered the rate's value as how much they needed to add to the function's output. Simon & Blume (1994) reported that additive reasoning does not lead to students' conceptualizing division quantitatively; instead, students use addition to replace thinking multiplicatively. This additive reasoning is also what Castilo-Garsow (2010) described when his student explained an interest rate as how much money to add to a bank account every year. If students use additive reasoning about in contexts that require them to use the idea of rate, this will affect how they approach learning and using the idea of derivative and what they imagine instantaneous rate of change at a point represents.

Theoretical Perspective

The Mathematics of Students

This study employs Radical Constructivist theories of learning (Thompson, 2000), taking the perspective that it is impossible to know another's thinking. Therefore, investigating student thinking aims to build models of students' mathematics (Steffe & Thompson, 2000) that may be used as an explanatory model for why students produce specific responses. The use of this approach to build an explanatory model of students' thinking can then be useful for describing the process of developing a productive meaning for instantaneous rate of change. I use 'meaning' the same way that Thompson (2013) uses it to describe mathematical meaning. It is the organization of an individual's experiences with an idea that determines how the individual will act. Meanings are personal, and they might be incoherent, procedural, robust, or productive, but individuals use these meanings to respond to mathematics tasks and make sense of and access mathematical ideas. For example, a person's meaning for derivative might only be associated with calculating limit of the difference quotient, while another's meaning for derivative involves the slope of a tangent line. Since meanings are personal, if one student writes a response similar to another student, it cannot be assumed that they both have the same meaning.

Conceptual Analysis

In a typical Calculus 1 course, instantaneous rate of change is introduced to students via the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)}$ (Stewart, 2013; Larson et al., 2006). One productive interpretation of the limit is the multiplicative relationship (the value of $f'(x)$) between a variation in a function's output ($f(x+\Delta x)-f(x)$) and a variation in the input ($(x+\Delta x)-(x)$) so long as the variation from the input value x is arbitrarily small ($\lim_{\Delta x \to 0}$). The convergence of this limit is what we call "instantaneous rate of change," which represents the relationship between two varying quantities with respect to one another's relative size of variation. To use a derivative value as a rate of change having some value m, one must imagine the input quantity varying while simultaneously imagining the output quantity varying m times as much as the input quantity's variation. This is the same meaning we might attribute to an average rate of change over a small interval, in that someone is imagining the necessary constant rate of change to achieve the same accrual in one quantity with respect to the accrual size of the other quantity. Additionally, a student also has to recognize that the output variation will be *essentially equal* to the actual variation since the quantity is not changing at a constant rate of change. In this case, what it means to be *essentially equal* is that the approximation of the variation in the output by assuming a constant rate of change will be so close to the actual variation that the difference between the two is imperceptible.

Methodology and Data Analysis

This study employed clinical interview methodology (Clement, 2000). Clinical interview methodology consists of an interviewer, a single student, and a camera to record each interview. During these interviews, the interviewer asks the student to engage in a mathematical task. These tasks intend to explore how students are reasoning about them. As the interview progresses, the interviewer creates models of the student's understandings and tests their hypotheses by asking questions and probing the student's thinking. Additionally, the interview makes no teaching moves since the purpose is to understand how the student is thinking about a particular topic.

Clinical interviews were conducted with 25 students at a southwestern university who were enrolled in a Calculus 2 course or were at the end of their Calculus 1 course. This study was part of a larger study that involved tasks designed to help investigate students' understandings of the derivative at a point and their meaning for rate of change. Only the first task is presented here.

This study was analyzed using Open and Axial Coding for moment-by-moment coding of students' responses and interpretations (Stauss & Corbin, 1998). Using the codes from each student, I conducted a thematic analysis (Clarke & Braun, 2013) across moments within each students' moments and across different students' moments. This thematic analysis aimed to identify and analyze the patterns of student responses to model the types of thinking that students were engaging in.

Results

In this study, each student was presented with a task about interpreting the derivative at a point and then performing a linear approximation [Figure 1].

Given that $P(t)$ represents the weight (in ounces) of a fish when it is t months old,
a.) Explain the meaning of $P'(3)=6$
b.) If $P(3) = 15$ and $P'(3) = 6$ estimate the value of $P(3.05)$ and say what this value represents.

Figure 1: The Fish Task

18 of the 25 students wrote an interpretation of $P'(3) = 6$ as representing instantaneous rate of change. By examining the written responses, two types of answers emerged from these 18 students; one stated the units as ounces per month, and the other stated units as ounces [Figure 2]. The other seven students did not include rate of change in their explanation. Some of these students only explained that it was the slope of the tangent line or the result of differentiation. However, they did not express another type of understanding that indicated they were associating the derivative at a point with instantaneous rate of change.

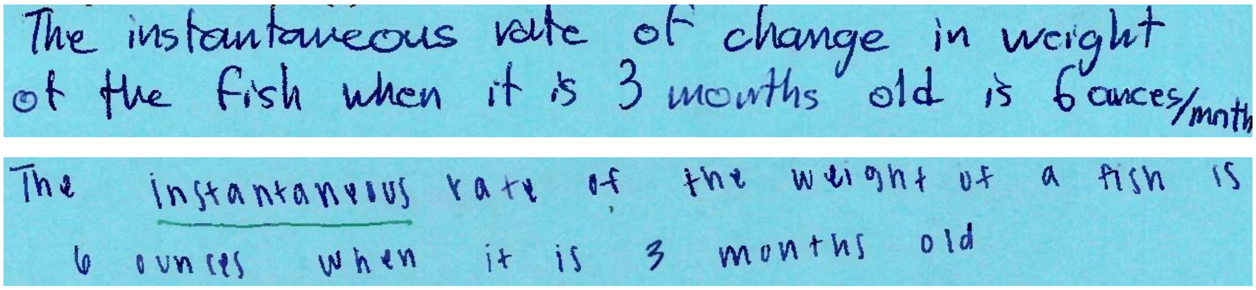

Figure 2: Students' Written Interpretation of $P'(3) = 6$

After students wrote their responses, each student was asked to explain their written statement. Even within the eight students who chose "ounces per month" as their units, different meanings emerged as students attempted to articulate their thinking. Similarly, the other ten students who chose "ounces" as their unit also expressed a variety of meanings despite having written similar responses. Figure 3 categorizes the meanings that students articulated when asked about what instantaneous rate of change meant to them in this situation. About one-third of the students chose ounces as their unit and this suggests that these students have not conceptualized a rate of change as involving two varying quantities. Some of these students conveyed that they interpreted the value of a rate as an additive difference in the weight of the fish (as opposed to a ratio between the two quantities of weight and time elapsed), which aligns with other researcher's findings that students confuse rate quantities with amount quantities (Byerley et al., 2012, Castillo-Garsow 2010).

Instantaneous Rate of Change with units as "Ounces per Month" (8 Students)

- 2 students described how weight and time would change in some interval of time
- 3 students stated "if the rate does not change then the fish will gain 6 ounces in 1 month"
- 1 explained that 6 ounces per month was found by finding the average rates of change over arbitrary intervals involving the time value of 3 months
- 1 described it as the average rate of change over the entire fish's lifetime
- 1 explained it as speed at a moment like "the reading of a speedometer"

Instantaneous Rate of Change with units as "Ounces" (10 Students)

- 3 students conveyed that the fish would grow 6 ounces in the entire 3rd month
- 5 said "if the rate does not change then the fish will gain 6 ounces in 1 month"
- 1 explained that the fish will grow 6 ounces every 3 months
- 1 described the average rate of change over the 3rd month.

Figure 3: Summary of Students' Explanations for Instantaneous Rate of Change

It is noteworthy that even if cases where students provided the same answer, they did not necessarily share the same meanings. Among this pool of 18 students, I identified eight different meanings for instantaneous rate of change. The following excerpts exemplify this finding.

Andrew was a 2nd-year student enrolled in a Calculus 2 course. He explained instantaneous rate of change as an amount of change in a 1-unit change in the time [Table 1].

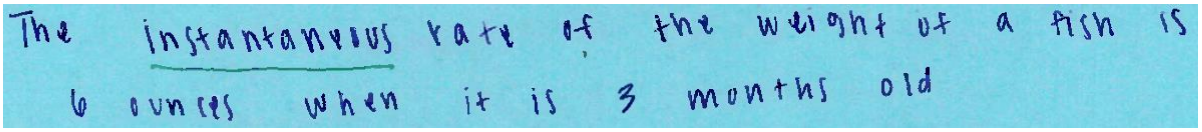

Table 1: Andrew's Explanation for the Derivative at a Point

And:	So it's at three months and it's growing by six ounces… it's weight increased by six ounces… that's how much it grew that month so for the third month it grew six ounces.
Int:	Like in that entire month?
And:	It gained 6 ounces.
Int:	So you're saying that if I have like a calendar month **Draws a calendar**, over the course of all this you're saying that the total change in weight is…
And:	6 ounces yeah
Int:	Okay, so what are the units for 6?
And:	Change in ounces.

Andrew interpreted instantaneous rate of change as "it's weight increased by 6 ounces" and "that's how much it grew that month" [Lines 1-2]. He also conveyed that he thought that the fish "gained 6 ounces" in the entire third month [Line 5]. This suggests that Andrew interpreted $P'(3) = 6$ as the difference in weight between the 4th month and the 3rd month, i.e., $P'(3) = P(4) - P(3)$. When asked about the units on 6, Andrew said it was a "change in ounces" [Line 10]. Andrew described the change as the difference in weight between two points in time. For him, this value of 6 did not describe a multiplicative relationship between variations in weight and time. Instead, it was how much weight to add to the fish's weight at the 3rd month to get the fish's weight at the 4th month. Andrew's description is akin to the kind of additive reasoning that Byerley et al. (2012) described. If we consider the verb tenses that Andrew employed, we can see that Andrew mostly used past tense verbs to describe the variation in the fish's weight. Andrew stated, "it's weight increased" [Line 1], "that's how much it grew… it grew 6 ounces" [Lines 2-3], and "it gained 6 ounces" [Line 5]. This suggests that Andrew might have thought that the values of the fish's weight were already there, and that $P'(3)$ was describing the variation in the fish's weight between the 3rd and 4th month. Andrew did not seem to be thinking about the value of this rate as describing the multiplicative relationship between the two varying quantities; instead, it appeared that Andrew was using chunky reasoning by thinking of an entire discrete chunk of change in the entire 3rd month. Additionally, Andrew never mentioned that time was passing; in fact, the only instance where he described an interval of time was at the beginning where he stated: "that month so for that month" [Line 2]. Andrew's explanations provide evidence that he was thinking about a chunk of time instead of time varying continuously.

Jerry was a 3rd-year Computer Science major at the end of his Calculus 1 course. Jerry explained instantaneous rate of change as an average rate of change [Table 2].

The instantaneous rate ~~that~~ the fish is gaining weight in ounces at specifically 3 months is 6 ounces/month

Table 2: Jerry's Explanation for the Derivative at a Point

Int:	So can you tell me what you wrote means to you?
Jer:	At 3 months, the rate that the fish is gaining weight is 6 ounces… so let's say the fish is gaining weight then at 3 months… so when you… it is like the average
Int:	Average?
Jer:	Yeah like the 3 is the amount of 3 months if it were to ummm like the average like a straight line graph **uses his finger to motion a straight line going up and to the right** at 3 months would be 6 ounces per… would be the rate like the average rate over those months
Int:	So instantaneous rate of change means average rate then?
Jer:	Like it is an average for those 3 months, and we are measuring it at that specific time of 3 months, like instantaneous.

Initially, Jerry struggled to articulate what he meant by instantaneous rate of change and ends by saying that it is an average [Lines 2-3]. Next, Jerry explained that he interpreted the value of the derivative of 6 as the average rate of change of the fish's weight over the entire fish's age up to the input value of 3 months [Lines 5-8]. Jerry's responses convey that he was thinking about a slope or a constant rate of change as he moved his hand to motion a straight-line graph when describing an average [Lines 5-6]. Afterward, Jerry confirmed that he interpreted instantaneous rate of change as referring to an average [Line 10]. Additionally, it appeared that to Jerry, the word '*instantaneous*' referred to an instant of time when he states "that specific time of 3 months, like instantaneous" [Lines 10-11]. One possibility for Jerry's explanation is that he may be recalling that the derivative involves the slope between two points ($\frac{f(b)-f(a)}{b-a}$) while forgetting the limiting process involved. Later on in the interview, Jerry continued to explain rate of change only in the context of a graph. This suggests that Jerry's image of change was primarily about the physical shape of a function's graph.

Randy was a 2nd-year in a Calculus 2 course explained instantaneous rate of change as a rate of change over a small interval of time [Table 3]

A. At 3 months old, the fish is growing at a rate of 6 oz per month

Table 3: Randy's Explanation for the Derivative at a Point

Ran:	Like the instantaneous rate of change, so like… that's how much it's changing by over a process of time. *Slides his hands to motion*
Int:	So like over the first three months it gained 6 ounces?
Ran:	No like…let's say like from…. like 2.9 to 3.1, you can assume like the average rate of change is 6, like from there to there it would keep changing by like 6 ounces per month.
Int:	So the average rate of change between 2.9 and 3.1 is always going to be 6?
Ran:	No, but I mean I'm just like trying to explain it cause it's not changing if time isn't changing

Randy did not initially write “instantaneous rate of change,” but when explaining what he wrote, Randy immediately said that it was “like the instantaneous rate of change” [Line 1]. As Randy explained what instantaneous rate of change meant to him, he articulated that he was thinking about “how much it’s changing over a process of time,” and as he said this, he slid his hands to express the imagery of this thinking [Lines 1-2]. As he continued explaining, Randy indicated that he was thinking about a variation in the input since he said “like from… 2.9 to 3.1…like from there to there” [Lines 4-5]. Randy clarified that he was not thinking of 2.9 and 3.1 as fixed values. Instead, he chose them because he was “trying to explain it cause it’s not changing if time isn’t changing” [Lines 8-9]. Here Randy articulated that he thought about quantities varying and associated the rate of change with these varying quantities. However, he did not explicitly say how the value of the rate of change corresponded to these quantities changing. It is important to note that Randy seemed to be thinking about quantities changing smoothly through intervals due to his choice in verb tense when he described a quantity changing. Randy repeatedly used “changing” when asked about what he meant; “changing by over a process of time [Lines 1-2], “keep changing by like 6 ounces per month” [Lines 5-6], and “it’s not changing if time isn’t changing” [Lines 8-9]. His repeated use of the word “changing” as well as his hand sliding motion, evidence that Randy was engaging in smooth reasoning by thinking about a change in progress.

Discussion

From the results of this study, we can see that students hold differing meanings about instantaneous rate of change. With these different meanings, one would expect that students would solve part b of the fish task (a linear approximation problem) in various ways as well. All but one student produced a correct solution, however, did not engage in the same ways of reasoning to do so (Yu, 2020). If a teacher did not inquire into how students produced their solutions, they would likely believe that their students understand instantaneous rate of change the same way they do. This also indicates that our current assessments on linear approximation are insufficient in evaluating students’ meanings about instantaneous rate of change.

The results of this study indicate that students do associate derivative with instantaneous rate of change, yet they may have different meanings about what instantaneous rate of change is. This study supports past findings (Zandieh, 2006; Park 2013) that students in a traditional calculus course might have unconventional meanings about derivatives and instantaneous rate of change. Therefore, teachers should consider what meaning they desire for their students to hold for derivative and instantaneous rate of change, and how students may be supported in building coherent and productive meanings. In particular, the results of this study suggest that Calculus teachers should attend to the meanings that students bring into their classroom about rate of change since many students consider rates as an amount to add instead of describing the multiplicative relationship between two varying quantities. Typically, most Calculus textbooks (Stewart, 2013; Larson et al., 2006) do not explicate a meaning for rate of change and likely assume that students understand what a rate of change is. I argue that teachers should intentionally spend time to aid students in developing a robust meaning for rate of change so that they can better understand a derivative as representing instantaneous rate of change.

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