Virtual Reality Applications of Math Education in Calculus: Visualizing Triple Integrals

Jonathan Troup
California State University, Bakersfield

This study details the embodied, symbolic, and formal reasoning of an undergraduate student when determining proper bounds for a triple integral problem within a virtual reality program called Calcflow. While the participant initially had difficulty inputting valid bounds, once he learned the appropriate constraints, he gradually connected various aspects of his symbolic reasoning with his embodied reasoning through progressive experimentation in Calcflow. Results suggest a virtual reality environment can help ease students' difficulties in visualizing three-dimensional shapes and thereby provide them the opportunity to develop deeper reasoning about these shapes and their corresponding equations.

Keywords: Bounds, Triple Integral, Virtual Reality, Embodied, Symbolic

Introduction and Literature Review

Research on difficulties with students' understanding of single variable calculus is extensive (e.g., Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Oehrtman, 2009; Tall & Vinner, 1981), and some of these difficulties might generalize to multivariable calculus (Dorko & Weber, 2014). Similar studies (e.g., Orton, 1983; Mahir, 2009; Nguyen & Rebellow, 2011) have indicated that visualizing a 3D object and setting up an integral may be the most difficult part of the solution (Sheikh & Fray, 2017). Students appear able to evaluate given integrals analytically but struggle to make connections between the integral bounds and the given 2D or 3D object. Thus, Sheikh and Fray (2017) asked students to use MATLAB to generate and view 3D solids, and to sketch projections, contours, and cross-sections of the solid, hoping these tasks would help students set up triple integrals to find volume for a given object. They found not many students mastered these tasks, particularly for cases where split integrals were required, or multiple bounding surfaces were involved. By utilizing Zazkis, Dubinsky, and Dautermann's (1996) Visualization-Analysis model, they reported that students' conversions between representations were problematic. Similarly, Kashefi, Ismail, Yusof, and Rahman (2012) wrote "for most students, imaging and sketching in 3-dimensions were the greatest difficulties that they encountered when doing non-routine problems in multivariable calculus" (p. 5537).

Habre (2001) reported that while computer programs assisted visualization, they did not develop students' ability to visualize without assistance. In contrast, Purnomo, Winaryati, Hidayah, Utami, Ifadah, and Prasetyo (2020) concluded a sequence of tasks completed in Maple helped students improve in multivariable calculus. Perhaps this distinction can be partially explained by the nature of the tool utilized. Zbiek, Heid, and Blume (2007) noted that the usefulness "of a cognitive tool becomes most apparent in the area of multiple representations when it is used to establish a 'hot link', that is, a dynamic connection between two representations such that an action taken in one representation is automatically reflected in the other" (p. 1174). The tool should also have a high degree of mathematical fidelity: the ability for the technological rendering to correctly match the underlying mathematical principle it is representing. They note it is important to encourage students both to play with the tool and reflect on the actions they took with it, to discover and develop mathematical meaning.

Kang, Kushnarey, Pin, Ortiz, and Shihang (2020) tested a program designed to help students visualize multivariable calculus concepts and found participants in the experimental group felt using a virtual reality (VR) program was more helpful to them than the control group's slides. However, the control group outperformed the experimental group. The authors suggested this may have occurred because the VR group learned without instructor guidance, while the slide group was paced. This hypothesis implies VR "cannot be a panacea for understanding advanced mathematics, and some form of intervention (whether via better implemented technology, or via in-person diagnosis of the student's ability) is required" (p. 58945). In a study on spatial visualization via VR, Herrera, Abalo, and Ordóñez (2019) claimed that VR helped students visualize mathematical concepts, especially for visualizing intersections between surfaces, noting increased average class score and reduced failure rate. Thus, usage of VR may help remove one of the primary obstacles to multivariable integrals-visualization of 3D objects. The current study thus attempts to establish whether a similar result holds in another mathematical context.

This study investigated whether utilizing a VR program as a visual aid in a series of directed tasks can help resolve difficulties in visualizing 3D graphs and finding bounds for a double or triple integral. This is an exploratory study designed to investigate some of the qualitative phenomena that learning math in VR is likely to elicit. Participants were asked to complete tasks involving double and triple integrals while utilizing *Calcflow*, a VR program. The research question is: What kinds of reasoning do students use as they attempt to find bounds of integration for a triple integral while working in a VR environment using multiple representations?

Theoretical Framework

The main principle of embodied cognition is that reasoning is inextricably linked to the physical environment. There are a variety of interpretations of this statement (see Alibali & Nathan, 2012; Lakoff and Nuñez, 2000; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Wilson, 2002), but for the author, this assumption carries with it the corollary that "learning is doing." Thus, speech, gesture, or technological actions all serve as evidence for the types of reasoning the participants might utilize. Thought processes are invisible, though we might speculate on potential thought processes based on physical action. Notably, in a VR environment, the lines between gesture and technological actions are significantly blurred.

Tall (2010) details a world of embodied mathematical thinking, which shares some overlap with the embodied cognition perspective. The embodied world is described by Tall (2010) as consisting of "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns…and other forms of figures and diagrams" (p. 22). The author thus considers the embodied world to consist of reasoning about geometric shapes, graphical phenomena, and connections of abstract mathematical phenomena to real-world contexts. Thus, the embodied world bears resemblance to both embodied cognition and *NCTM*'s (2009) definition of geometric reasoning. The symbolic world, as described by Tall, consists of "practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number)" (p. 22) So, the author characterizes the symbolic world as consisting of reasoning about algebraic or algorithmic processes, similar to *NCTM*'s definition of algebraic reasoning (2009). Finally, Tall writes that the formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22) Thus, the author considers the formal world to consist of structural relationships between mathematical concepts, and so includes proofs, theorems, conjectures, and formal logic. This study investigates how participants engaged with each of these worlds.

Method

Participants consisted of two undergraduate students who had previously completed the entire calculus sequence (Calculus I-IV) at a southwestern university and are referred to as Mike and Connor in this paper. Connor had completed the calculus immediately before the interview took place, while Mike was a senior who had completed the calculus sequence two years before the interview. Each participant was interviewed separately. This paper focuses on Connor. A one-hour task-based interview was conducted, immediately after the spring semester had concluded. In this interview, participants were to use *Calcflow* to determine proper bounds for a given shape. *Calcflow* allowed participants to enter bounds via a virtual keyboard, and generated the surface corresponding to those bounds. Thus, *Calcflow* requires symbolic input and displays corresponding embodied output. The left side of Figure 1 shows the bounds already entered when the program starts, and the corresponding surface as a blue graph within a transparent cube. This cube can be picked up, moved, dilated, and rotated. Participants could also summon virtual chalk to draw freely in the virtual environment in full 3D, though this action generates no other embodied output. The right side of Figure 1 shows such equations drawn with the chalk.

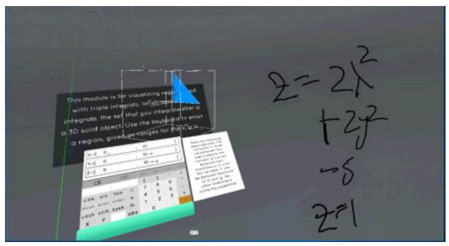

Figure 1: A snapshot of Calcflow

Interviews were recorded with screen-capture software to record participants' interactions within the virtual environment, and an external camera to record their gesture and other movements within the real physical world. These videos were analyzed, with particular attention paid to which of the three worlds participants were utilizing at any given point. All recorded speech, gesture, and technological actions were transcribed to an Excel spreadsheet, and each line coded according to which world or worlds the participant appeared to be utilizing. Summaries were then written detailing the participants' progression through each task. Finally, referencing the Excel spreadsheet and raw data as needed, I performed a cross-case and within-case (Merriam, 2009; Patton, 2001) analysis on these written summaries.

Results and Discussion

The interviewer started the interview by helping Connor fit the headset and introducing the VR controls. Then, the interviewer read the following question: Find the bounds for $\iiint_E xydV$, where E is the solid bounded by $z = 2x^2 + 2y^2 - 5$ and $z = 1$.

Connor made the connection between the symbolic expression $z = 1$ and its corresponding embodied planar shape immediately and recognized $z = 1$ as the upper bound. He observed:

> And, so I'm writing out the $x, y,$ and z plane, and I know that it's bounded at least (Creates dotted lines in VR at $z = 1$ above the y-axis and above the x-axis) by z equals 1. So I made a plane (Moves hand over the dotted lines, back and forth a few times) at z equals 1 as the upper part of it.

When the interviewer asked Connor what the resulting shape should be, he said "paraboloid." Connor began sketching a graph by hand by plotting individual points, then instead wrote several symbolic algebraic equations, including what appears to be $2 - 5 = 2x^2 - 2y^2$. Connor divided this equation by 5 and then erased it. When the interviewer asked him about his algebra, he said, “I tried moving the 5 to the other side, but that’s for an ellipse.**”** This attempt at symbolic reasoning suggests Connor had some trouble finding the intersection of the two bounding equations. The interviewer asked about the cross-section of the bounds, and after a short pause, Connor answered “$2x^2 + 2y^2$? That’d be like a circle.” Connor wrote $2x^2 + 2y^2$ and claimed after a longer pause, “so that is a circle with radius $\sqrt{2}$, or something like that”. Connor may have thought $2x^2 + 2y^2 - 5 = 1$ would become $2x^2 + 2y^2 = 5 - 1 = 4$ when combining constants, so this mistake may have been just a sign error. Connor drew a circle of radius $\sqrt{2}$ on the $xy -$ plane, suggesting he was making connections between his symbolic and embodied reasoning regarding the shape of the cross-section, but not its positioning. So, at the beginning of the interview Connor appeared to have difficulty coordinating aspects of his symbolic and embodied reasoning when multiple bounding surfaces were involved, as in Shiekh and Fray (2017). Notably, he was sketching by hand at this stage, so he did not have the benefit of immediate computer-generated output, which might explain some of his difficulty. Similar partial correspondences are evidenced by his completed sketch, as he identified $z = 1$ as the upper bound but drew an inverted paraboloid and reflected it across the xy −plane. (Figure 2).

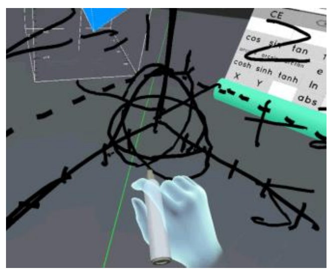

Figure 2: Virtually hand-drawn paraboloid and its reflection across the $\boldsymbol{xy}$ −plane

The interviewer noted *Calcflow* fixes the order of integration as $dzdydx$, so the x bounds must be constant, the y bounds can only involve the variable x, and the z bounds can involve x and y. Connor still input many bounds that violated these constraints, perhaps because *Calcflow* rendered output even for some of these expressions. For instance, the bounds $x^2 \leq x \leq 2, 0 \leq y \leq 2$, and $1 \leq z \leq -1$, generated output that Connor called a “cube” (see Figure 3), and $x^2 \leq x \leq 2$, $2x^2 \leq y \leq 2$ created a blue graph that showed a sequence of vertical blue lines stretching back and to the right (see Figure 4).

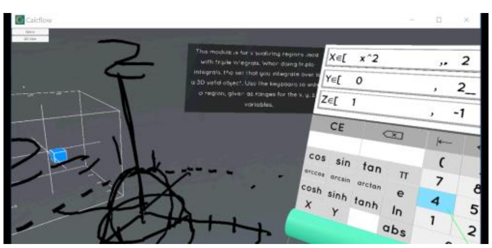

Figure 3: Self-referencing bounds

Figure 4: Infinite blue graph

This behavior reduces *Calcflow*’s mathematical fidelity (Zbiek et al., 2007), which can cause a learner to have difficulty distinguishing which aspects arise from the mathematics and which arise from the tool. Connor avoided further self-referential bounds only after a second warning from the interviewer. He entered $2 \leq x \leq -2, 2x^2 \leq y \leq -2$, and $1 \leq z \leq -1$. This input generated a rectangular prism with a parabolic cylinder cut out (see Figure 5). Connor observed:

2 is less than x is less than negative, I got a parabola. That's pretty cool. Um, so that's $2x$. Okay then I have, understood, okay. So the y part, so this, this part right here (Waves cursor over trough shape along the axis stretching to the right) must be the y-axis.

Connor thus connected his symbolic input to the embodied output. By changing the lower y bound to $2x^2$ and observing the resulting graph, Connor inferred the parabola must lie along the $y-$axis. He extended this connection between his embodied and symbolic reasoning, saying, "to make it a circle, I'd have to put the circle equation into y maybe." Connor thus entered the bounds $2 \le x \le -2$, $(2-x^2)^{\frac{1}{2}} \le y \le -2$, and $1 \le z \le -1$, producing a blue graph that Connor called "something kind of circular" (Figure 6).

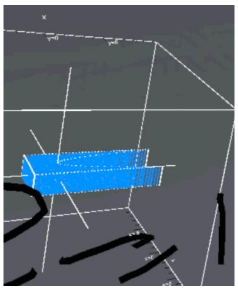
Figure 5: Connor's "parabola"

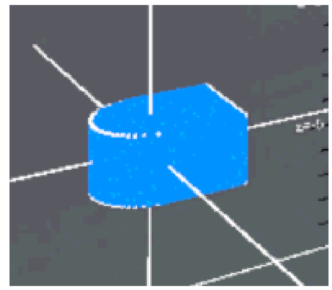
Figure 6: "Kind of circular"

Note he continued this pattern of making embodied observations and changing the symbolic bounds to address that observation. He said, "The back half is not circular though, so I need to figure out how to fix that. The back half is straight." Connor typed $(2-x)^{\frac{1}{2}}$ into the lower y bound and $-(2-x^2)^{\frac{1}{2}}$ into the upper y bound, which resulted in a full cylinder output.

> Hey there we go. (Looks at blue graph which now displays a full cylinder instead of a half cylinder. And then you need to shift it. (Changes lower y bound to $(2-x^2)^{\frac{1}{2}} - 5$. Looks at blue graph, which now displays a rectangular prism with cylinder shapes cut out of either end. Highlights upper y bound and upper z bound.) Okay. (Changes upper y bound to $-(2-x^2)^{\frac{1}{2}} - 5$, looks back at graph which is now a cylinder centered on the y axis (not the origin), but pointing in the z direction.)

Thus, Connor connected the symbolic -5 to a graphical embodied "shift". When he changed the upper $z-$bound from 0 to $-(2x^2+2y^2-5)$, he noticed that "when I put in the negative it just reflected over the axis," as seen in Figure 7.

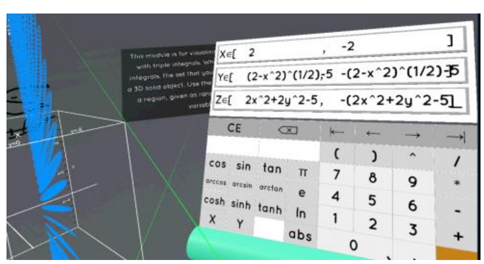
Figure 7: Reflection over axis

Afterwards, Connor observed that he had added a -5 onto the y bounds. He removed these and seemed excited by the result (see Figure 8). The bounds at this stage were $2 \le x \le -2$,

$(2-x^2)^{\frac{1}{2}} \leq y \leq -(2-x^2)^{\frac{1}{2}}$, and $0 \leq z \leq 2x^2 + 2y^2 - 5$.

Oh, and then I put a minus 5 and a plus 5. Let me delete this see what happens to (Backspaces -5 out of lower y bound) Ooh okay. (Looks at blue graph which looks like a paraboloid on one side but fanning upward infinitely on the other side. Backspaces -5 out of upper bound. Looks back at blue graph, which was an upward facing paraboloid with a cylinder on top, as shown in Figure 9.) Now we're getting somewhere.

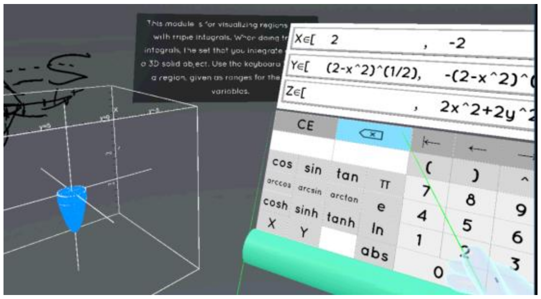

Figure 8: Cylinder on paraboloid

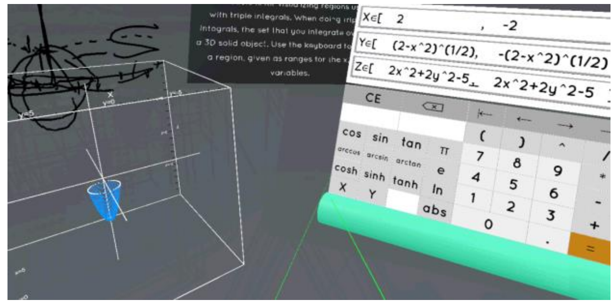

Figure 9: The very thin line

Connor deleted the -5 out of the upper z bound as well, which resulted in a cylindrical output. He then changed the lower z bound to the same as the upper z bound, and again connected his "thin line" embodied output to the identical expressions for z in his symbolic input:

I got a very thin, Oh! I understand why. It's doing a very thin line (Looks over at blue graph.) because I made the z, and it's the same thing. (Mouses cursor over lower z bound. Moves cursor back and forth between upper and lower z bounds)

The interviewer asked whether this shape matched his sketch, and Connor laughed and said no.

Interviewer: What's wrong with it?

Connor: So, I thought it was more spherical in shape, but this is definitely like, this makes more sense that it's paraboloid shaped.

At the end of the task, the bounds read $2 \leq x \leq -2$, $\sqrt{2-x^2} \leq y \leq -\sqrt{2-x^2}$, and $2x^2 + 2y^2 - 5 \leq z \leq 0$. Although at this stage of the interview the upper z bound was not correct, recall Connor recognized the plane $z = 1$ as an upper bound at the beginning of the interview. Circular equations bound x and y; they just use an incorrect radius. The $x-$ and $y-$ bounds are backward, likely because *Calcflow* makes no observable distinction in its rendering based on this ordering. So, Connor came close to correct bounds through experimentation, observation, and coordination of symbolic and embodied expressions in *Calcflow*.

Qualitative Post-Interview

Connor independently described the experience overall in a way that reads like a natural summary of the embodied cognition theoretical perspective, saying:

[*Calcflow*] allow[s] you to, um, play around, and see the shapes you're getting towards as you get to your final function that you want to see. And it gives you a sense of how each one of the different variables and their placement in x, y, or z is helpful in, you know, making the graph the way it is.

Similarly, Connor contrasted the difficulty of making sense of 3D figures on paper with the comparative ease of making sense of these same figures in a VR environment. He explained:

When it's depicted in the calculus book that we use, it uses, like, shading in lines, but it's difficult to find the difference between, um, let's say a plane is coming toward you and

> sloping down. It's hard to see whether they're saying that that is an edge or whether the lines are coming toward you so close that they're not an edge, but it's slightly sloping up, and so this, this allows me to grab it and actually, you know, manipulate it and move it around.

Connor remarked it was easier to draw 3D shapes in *Calcflow* than on paper, probably because to do so in *Calcflow* one need only move their hand through the air in the same way they would as if they were tracing an actual 3D object of the same shape. On the other hand, Connor seemed discouraged from experimenting by the input mechanism, asking the interviewer on one occasion to verify he had the correct bounds before typing them.

Concluding Remarks

The results of this investigation highlighted that Connor developed connections between his symbolic and embodied reasoning with the help of *Calcflow*, that he was misled at times by some breaches in mathematical fidelity, and that he felt he could draw, manipulate, visualize, and comprehend 3D objects more easily in *Calcflow*'s VR environment than on paper. By gradually connecting aspects of embodied and symbolic reasoning, Connor was able to come close to finding correct bounds. *Calcflow* likely helped him connect these two forms of reasoning by requiring symbolic input and producing corresponding embodied output, and aided by this "hot link", Connor eventually appeared to attend to both representations in tandem. This dynamic enabled Connor to experiment and determine how the graph changed in response to the input bounds, and it further enabled him to test the validity of his inferences and generalizations immediately. Like the coordination between algebra and geometry seen in Zazkis et al. (1996), the coordination between embodied and symbolic reasoning itself may have helped drive Connor's progress. Early on, Connor appeared to have difficulty attending to multiple representations, while later he discovered and leveraged connections between them. In the qualitative post-interview, Connor claimed it was easier to visualize the 3D shapes with VR than with other 2D representations, even interactive ones, suggesting both that such visualization might be one of the primarily obstacles in determining these bounds and that *Calcflow* can help alleviate this burden. There were no identified instances of Connor utilizing formal reasoning as described by Tall (2010). Given Connor's repeated struggles avoiding self-referencing bounds, the author agrees with Zbiek et al. (2007) that high mathematical fidelity in a program is desirable, and Kang et al. (2020) that "technology itself cannot be a panacea for understanding advanced mathematics, and some form of intervention (whether via better implemented technology, or via in-person diagnosis of the student's ability) is required" (p. 58945).

Most of Connor's praise seemed to revolve around the embodied graphical output of *Calcflow*, and most of his criticism around the symbolic input mechanism, which proved difficult to use. Thus, future research could focus on whether a similar VR program that improves on this aspect would help further. For example, *Geometer's Sketchpad* allows direct manipulation of the shapes, as well as their corresponding equations. Troup (2018) suggests that moving from embodied to symbolic reasoning can offer different affordances than moving from symbolic to embodied reasoning. Thus, it may be beneficial to allow for similar manipulation of these shapes in *Calcflow*, so that students can change the embodied representation to see changes in the corresponding representation, to allow for a bi-directional "hot link". One could also find and investigate tools that further leverage the affordances of VR, such as those that would allow investigation of cross-sections obtained from slicing a solid, or various projections.

References

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*, *21*(2), 247-286.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, *15*(2), 167-192.

Dorko, A., & Weber, E. (2014). Generalising calculus ideas from two dimensions to three: How multivariable calculus students think about domain and range. *Research in Mathematics Education*, *16*(3), 269-287.

Habre, S. (2001). “Visualization Enhanced by Technology in the Learning of Multivariate Calculus”, The International Journal of Computer Algebra in Mathematics Education, Vol. 8, No. 2, pp 115-129.

Kang, K., Kushnarev, S., Pin, W. W., Ortiz, O., & Shihang, J. C. (2020). Impact of Virtual Reality on the Visualization of Partial Derivatives in a Multivariable Calculus Class. *IEEE Access*, *8*, 58940-58947.

Kashefi, H., Ismail, Z., Yusof, Y. M., & Rahman, R. A. (2012). Fostering mathematical thinking in the learning of multivariable calculus through computer-based tools. *Procedia-Social and Behavioral Sciences*, *46*, 5534-5540.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, *40*(2), 201-211.

Merriam, S.B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco: John Wiley and Sons.

National Council of Teachers of Mathematics. (2009). *Focus in High School Mathematics: Reasoning and Sense Making*. Reston, VA: National Council of Teachers of Mathematics.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences*, *21*(2), 287-323

Nguyen, D. H., & Rebello, N. S. (2011). Students’ difficulties with integration in electricity. *Physical Review Special Topics-Physics Education Research*, *7*(1), 010113.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1-18.

Patton, M. Q., 1991. *Qualitative evaluation and research methods*. (3rd ed.). Sage Publications Inc., Thousand Oaks, CA, London, New Dehli.

Shiekh, T., & Fray, R. (2017). Visualization, representations, and evaluation of volume integrals in calculus. In Penlington, T., & Chikiwa, C. (Eds.) *Proceedings of the 23rd Annual National Congress of the Association for Mathematics Education of South Africa*, (pp. 271-288. Port Elizabeth.

Stewart, S., Troup, J., & Plaxco, D. (2019). Reflection on teaching linear algebra: Examining one instructor’s movements between the three worlds of mathematical thinking. *ZDM Mathematics Education*, *51*(7), 1253-1266.

Troup, J. (2018). Developing Reasoning about the Derivative of a Complex-Valued Function with the Aid of Geometer's Sketchpad. *International Journal of Research in Undergraduate Mathematics Education*, 1-24.

Wangberg, A. "Fostering Student Discovery and Conjecture in Multivariable Calculus," Journal of Humanistic Mathematics, Volume 10 Issue 2 (July 2020), pages 315-347. DOI: 10.5642/jhummath.202002.15. Available at: https://scholarship.claremont.edu/jhm/vol10/iss2/15

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic bulletin & review*, *9*(4), 625-636.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D_4. *Journal for Research in Mathematics Education, 27(4)*, 435-457.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. *Second handbook of research on mathematics teaching and learning*, *2*, 1169-1207.