Student Understandings of Properties of Linear Transformations

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Student understanding of linear algebra concepts is a growing research area. This study explores students' conceptualization of linear transformations and the various ways in which they use such conceptualization to reason through linear transformation problems after they have completed a linear algebra course. Three students participated in a task-based interview. Through analyzing interview data using a grounded theory approach, emerging themes were found indicating that students' exposure to linear transformations in other courses and the nature of these experiences impacts how they further conceptualize linear transformations. Notably, the way that the participants were engaged with linear transformations within their other courses for their major seemed to influence their use of geometry, algebra, and proofs in determining whether a given transformation is linear. One implication of this study is a need to engage students with more real-life applications of linear transformations in linear algebra courses.

Keywords: Linear algebra, linear transformations, representation, reasoning.

Introduction

Linear algebra is a course that many students are required to take as a prerequisite for other mathematics courses as well as courses in other disciplines. It is important to consider with what conceptions students leave their linear algebra courses to determine how they will view and interact with linear algebra concepts in future courses. Knowing students' potential conceptions can inform instructors of both linear algebra and other courses that utilize linear algebra topics on how to develop research-based learning opportunities on these topics for improved understanding.

There has been a large amount of research investigating student reasoning in the areas of span, linear independence, eigenvectors, and eigenvalues. (See Stewart, Andrews-Larson, & Zandieh, 2019 for a recent overview.) However, these studies focus on mostly students' geometric reasoning of these topics (Stewart, Andrews-Larson, & Zandieh, 2019). To have a robust conceptualization of linear transformations, a student needs to perceive them as more than just geometric movements. "Linear transformations are functions from one vector space to another, often $\mathbb{R}^n$ to $\mathbb{R}^m$, with particular linearity properties (they preserve addition and scalar multiplication)" (Bagley et al., 2015, p.36). In other words, a robust conception that connects the geometric with these algebraic and abstract descriptions of linear transformations is important. Other studies have shown that students need more opportunities to make connections between their geometric descriptions of linear transformations and the algebraic representations of transformations (Bagley et al., 2015; Andrews-Larson et al., 2017). Thus, there is still a need for more research to gain deeper knowledge on student understanding of properties linear transformations: what they represent (for students) and how they can be used (Stewart, Andrews-Larson, & Zandieh, 2019). By determining more about this understanding, the Mathematics education community can further the design practices of linear algebra topics to incorporate existing student conceptions and challenges that would move these conceptions forward. The purpose of this research is to further investigate the conceptions and reasoning used by students who have taken a linear algebra course to utilize linear transformations. My research questions are: What conceptualizations do undergraduate students have of linear transformations? In particular, what mathematical reasoning do students implement in problem-solving situation related to linear transformations?

Literature Review

How to implement geometric ideas about linear transformations in the classroom has been a focus of research, and studies have shown benefits for student understanding by having geometric representations and application problems in related lessons. Bagley, Rasmussen and Zandieh (2015) considered students conceptions of linear transformations as functions. They specifically explored ways in which students made connections between the concept of function they learned in previous algebra courses with the concept of transformations they learned in their linear algebra course. Bagley et al. (2015) described student conceptions that were not necessarily productive as pseudostructural conceptions (Sfard, 1991). The authors stated this type of conception seemed to cause students to struggle to identify the need for vector notation or how it was involved in linear transformations. Andrews-Larson, Wawro, and Zandieh (2017) also emphasized that students' conceptions of linear transformations needed improved flexibility, and the authors offered a hypothetical learning trajectory to help build this flexibility.

Stewart, Andrews-Larson, and Zandieh (2019) provided a summary of existing research in Mathematics education pertaining to linear algebra and gave directions for future research. They found that in the research regarding student reasoning and understanding, there was significant work considering geometric representations in linear algebra, proofs, and eigen theory. However, the authors pointed out that there was less research regarding students' algebraic reasoning with linear transformations and other properties of linear transformations. They called for research in the teaching of linear transformations and claimed that further research needs to be conducted to determine what claims from these studies can be further generalized. In knowing this, research in the teaching of linear algebra can grow and change.

Turgut (2019) does specifically study student understanding of linear transformations and explores student reasoning regarding matrix representations using geogebra, and the implications for teaching linear transformations. This study focused on different matrix and geometric representations of linear transformations and investigated how students interacted with technology modelling linear transformations. Turgut (2019) found that students had a difficult time transitioning to matrix representations of linear transformations from other forms of representations. This, the most current research on linear transformations available to my knowledge, indicates a need for further exploration of student understanding that encompasses contexts outside of the use of technology. The study I present in this paper aims to consider student understanding of linear transformations in a different way from Turgut (2019) by exploring students' reasoning involving different representations of linear transformations, as matrices, as functions, geometrically, and so forth. Furthermore, this study differs from the existing research by incorporating grounded theory methodology to explore not only students' differing conceptions and use of representations but also to investigate their ways of reasoning with representations after completing a linear algebra course.

Methodology

Due to the limited nature of existing studies that capture students' conceptions of linear transformations and students' uses of properties of linear transformations after students complete a linear algebra course, this study is designed to develop a substantive theory (as opposed to "grand" theory) to expand on the existing research. The grounded theory methodology (Merriam & Tisdell, 2015) guided this qualitative inquiry to address the research questions: What conceptualizations do undergraduate students have of linear transformations? And, what mathematical reasoning do students implement in problem-solving situation related to linear transformations? The interview data from three students were analyzed using themes and a constant comparative method and a core category emerged in the data, building to a substantive theory.

Theoretical Framing

I utilized the theoretical framework of constructivism that assumes students construct their knowledge through their experiences in various settings and with others (Crotty, 2015). Centered in this perspective a conception is defined as "a theoretical construct within 'the formal universe of ideal knowledge'; the whole cluster of internal representations and associations evoked by the concepts" (Sfard, 1991, p. 3). This theoretical construct together with Sfard's (1991) duality principal of conception shaped my overall design of the study. Students' conceptions at the time of the interview were assumed to be displayed through their utterances including written outputs and gestures. During the constant comparative stage of data analysis, I operationalized Sfard's (1991) operational and structural conceptions descriptions to understand the ways in which participants performed "processes, algorithms, actions" (p. 4) on the notion of linear transformations as well as their (flexible) use of representations of linear transformations.

Methods

In this grounded theory study, semi-structured in-depth interviews were conducted to explore participants' current conceptions of linear transformations by asking them to identify whether given transformations were linear or not. Participants were asked to solve linear transformation problems and provide their reasoning. Interviews were 30-90 minutes in length and were audio and video recorded and transcribed after the interview. The interviews were conducted during Spring 2020 semester at a mid-sized university in the Rocky Mountain Region of the United States. At this university physics and mathematics majors are required to take the linear algebra course offered by the Mathematics Department. The department offers only one linear algebra course every semester and the course is matrix-oriented that covers the core topics including eigenvalues and eigenvectors with limited focus on proofs. The linear algebra course serves as a pre-requisite for several undergraduate courses including abstract algebra, numerical analysis, and modeling courses.

Students from mathematics courses for which linear algebra is a prerequisite were invited to participate in the study. To select participants, students were given a survey with questions about their year in school, when they took linear algebra, their mathematics course history, their majors, available times for interview, and their preferred pseudonyms and pronouns. Three participants, Perry, Jed, and Summer (preferred pseudonyms) were selected for the study according to their availability for an interview, their year in school, mathematics course history and major to include a varying range of experiences. All three students participated individually in one interview. Perry (pronouns:they/them) was a junior secondary education mathematics major and took linear algebra in Fall 2019 semester. Jed (pronouns: he/him) was a senior physics major and took linear algebra in Fall 2018 semester. The third participant, Summer (pronouns: she/her), was a senior mathematics major with an emphasis in applied math and applied statistics and took linear algebra in Spring 2017 semester.

Data Collection and Analysis

Each participant was given a series of transformations to consider in matrix, function, graphical, and descriptive formats. These included 2x2 matrices, linear functions, and geometric depictions of images transformed on a set of axes. The interview included a variety of the types of interview questions listed in Zazkis and Hazzan (1998) to address the research questions. For example, I used "give an example and non-example of a linear transformation" tasks.

Within a grounded theory approach, open and axial inductive coding processes are fundamental to develop a substantive theory. This study followed these processes to analyze the interview transcripts. In the first cycle of coding, transcripts were categorized according to their relevance to each research questions. There was some overlap where participant responses applied to both research questions. Open descriptive coding was then used to assign words and phrases to these transcript "chunks". In the second cycle of coding, axial coding was used to relate these chunks of transcript based on their open codes. Through this axial coding, themes emerged. Some of these themes were about students' conceptions that were reported in existing research studies such as geometric and algebraic interpretations. New themes that emerged were related to the students majors and other courses they had taken.

Results

The analysis of the interview data indicates that three participants had three different conceptions of linear transformations and approached the problems in different ways. There were common themes in some of their approaches to individual problems, and the type of notations they used including geometric examples, algebraic examples, and informal algebraic proofs. In addition, there were commonalities among participants' reasoning and the ways in which they made connections to their coursework and majors in such reasoning instances.

Insight into participants conceptualizations of linear transformations

Perry seemed to have an evolving conception of linear transformations. Jed's conception seemed to be more robust, and Summer's conception seemed to be process oriented (Sfard, 1991). Perry's interview began with them struggling to recall what a linear transformation was. They relied on their concept of lines and transformations of lines specifically. As the conversation continued, Perry began to recall some of the types and properties of linear transformations:

Perry: Like vectors always start from the origin. Or they act like they always start from the origin so reflecting them over a line doesn't really do anything to the vector.

Interviewer: Can you draw what you are talking about?

Perry: Yeah I guess I could also just flip it over a line like this and have it be there (draws picture reflecting a vector over line y=x). Oh it's like we can alter the magnitude of the vectors…so we can multiply this vector by two and get wow ok, bigger vector.

Recalling what you can do with a vector geometrically and the realization of vectors "starting at the origin" helped Perry to reconstruct their understanding of linear transformations and properties such as the zero (vector) has to be mapped to zero (vector). They often updated their definition along the way. The following exchange occurred when they were unable to come up with a matrix representation when asked whether $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ |y| \end{bmatrix}$ is a linear transformation:

Perry: Because I am getting the idea that I need to write a matrix to have a linear transformation, which breaks my definition, in this case at least.

Interviewer: What's giving you that idea?

Perry: I could do it for all the other ones. Why can't I do it to this one? I don't like it.

Jed was able to recall almost immediately what a linear transformation was and used a formal mathematical definition of linear transformations. He said, "I mean algebraically I would say that it is a, any function t that satisfies this [pointing at the last line in Figure 1 is linear. And then generally if we're acting on an n dimensional space, these are represented by n by n matrices."

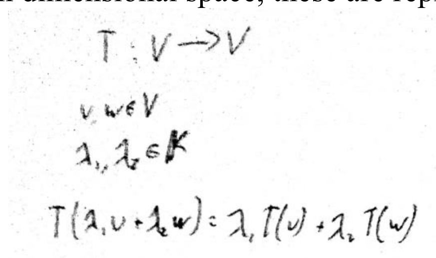

Figure 1. Jed's definition for linear transformations

Jed also gave a correct example and non-example of linear transformations and was able to articulate why these did and did not meet the definition for linear transformations. Overall, Jed seemed to have a more connected conception of linear transformations.

Summer had difficulty recalling linear transformations. She referred to her statistical knowledge and she said, "Coming from my stats background…linear is usually what I think of with like, linear regression. So relating things to one another in just like, one degree, not necessarily multiple, that's where my mind goes." Without further prompting she was unable to produce a working definition for linear transformation, so without indicating that it was the definition for a linear transformation (to determine if she would recognize it as such) I provided her with a copy of "rules" and asked her to determine if the transformations aligned with these rules. She did not say anything at this point about whether she recognized this to be a linear transformation, although towards the end of the interview she said she had wondered if that was the case. Overall, Summer's conception of linear transformation was limited to processes, rather than being abstract.

Insight into student reasoning in problem solving

As Perry refined their definition of linear transformations, they relied less on geometric interpretations and the more they used algebraic and matrix notations and their arguments became mathematically formal with resemblance of a mathematical proof. When presented with matrix representations of linear transformations they seemed to be hesitant. However, they conjectured that multiplying that matrix by a vector would transform the vector. They initially did not recall how to multiply matrices, but after giving it some thought, they correctly multiplied the given matrix by a generic vector to determine how the matrix transformed the vector both algebraically and geometrically. In this process their work resembled the notation and reasoning used in formal mathematical proofs.

Jed gave examples and non-examples of linear transformations as discussed in the previous section. He was able to use his formal definition to prove using algebraic methods whether a transformation was linear or not and give a matrix representations and determine the type of transformation geometrically without using specific examples of vectors and was able to give geometric interpretations when prompted. Jed also noted that he felt his knowledge of linear algebra was more from his physics (major) courses rather than the linear algebra course he took:

Jed: [M]ost of my knowledge comes from the functional analysis that is used in quantum mechanics. So we use a lot of stuff with basically finding the eigenfunctions and eigenvalues of differential operators. And then another decent chunk of it comes from knowledge from Tensor analysis and differential geometry that I did with an independent study.

Summer used a combination of geometric ideas and algebraic ideas to determine whether the transformations were linear or not but did not use formal algebraic notation. She also did not view matrices as representations of linear transformations. When presented with the matrix representation $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ of a linear transformation she attempted to construct her own definition for that notation, referring to the first column as the "input vector" and the second column of the matrix as the "output vector".

Summer: Oh, okay. So if it's doing the transforming, what I would assume is that your first column is your input vector essentially, and then your second column, the on the right hand side is your output vector. So we'd have a linear transformation of multiplying that vector by two would be my assumption, and that's also coming from thinking of matrices from a neural network standpoint. Because we've been doing that with our project in Math 437. And that's how I've kind of been remembering matrices is lots of like input-output kind of stuff.

With significant prompting from the interviewer, she was able to recognize that if the matrix itself was the transformation, then matrix multiplication would be how it transformed something a vector or any matrix with exactly two rows. Additionally, the excerpt from Summer's interview above provides another example of experience in other, more recent coursework having an influence on a participant reasoning with linear transformations.

Discussion

Both Summer and Perry appear overall to have pseudostructural conceptions (Sfard, 1991) of linear transformations. Perry demonstrating a pseudostructural conception of matrix representations of transformations aligns with the results of Bagley et al. (2015) This seemed to impact their ability to use matrix representations to determine what the transformation does and whether it is a linear transformation. These preliminary results also show a tendency of these students to rely on concepts from their most recent coursework rather than recall their linear algebra course specifically and for Jed and Summer this was specifically courses related to their major and interest areas. Summer often drew on her statistical knowledge to reason through linear transformations until she was given a different definition of transformation. Jed specifically described his experience in physics courses as having a large impact on his knowledge of linear algebra and how he was able to reason through these linear transformations.

Both Perry and Summer are math majors, and both utilized geometric interpretations more frequently than Jed. Summer however was more likely to use examples rather than proof techniques which seems to align with her applied math major. She has not taken proofs-based courses that Jed and Perry have, and considered examples enough to support her arguments. Perry would use generic examples as can be found in formal mathematical proofs and used formal algebraic notation. Jed was able to answer these questions about linear transformations seemingly without much difficulty. He had a significant amount of recent coursework that required ideas from linear algebra specifically. He also had coursework involving formal proofs of linear algebra related concepts. These experiences seemed to lead to his default approach of formal algebraic proofs. He demonstrated flexible switch between geometric and algebraic explanations when asked. Overall, Jed seemed to have more than a pseudostructural conception. This theme aligns with results of Karakok (2019) in that students' conceptions continue to evolve particularly in their future courses. Karakok (2019) noted that physics students' conception of eigen-values and -vectors showed changes from completion of a linear algebra course to completion of a series of quantum mechanics courses.

Conclusion

The purpose of this research study was to explore students' conceptions of linear transformations and their reasoning during problem-solving situations that involved properties and representations of linear transformations. The themes observed in this study lead to the substantive theory that at this university the conceptions of students whose main exposure to linear transformations has been the introductory linear algebra course are likely to remain at most at the pseudostructural level without further coursework. This prevents them from drawing connections between the matrix representations of linear transformations and their geometric implications. It seems that the ways in which students are exposed to linear algebra concepts beyond the introductory linear algebra course impact their conceptions of linear transformations and the ways that they reason in problem-solving situations with linear transformations.

This study is limited in scope due to the small sample size of three students. However, the findings have many implications specific to this university and for future research. The results of this study could be used to inform instruction not only of linear algebra but also of other mathematics courses. As prior research studies suggest that pseudostructural conception of linear transformations is problematic for students of all majors, more research should be done in alterations that can be made to help students restructure their conceptions of linear transformations. An alteration similar to that presented in Andrews-Larson et al. (2017) might be implemented to improve the flexibility of the students' conceptions. Because this course is required for many students of different backgrounds and majors, attention should be placed on making sure all representations of linear transformations are being addressed and that connections between these representations are made clear to students. In follow up courses within certain majors, adjustments may need to be made in understanding what type of conceptions of linear transformations those students may have when entering the course. When a robust, coherent conception of linear transformations is the goal for students in a mathematics program, it is important that students are given opportunities to make connections between and among many representations and contexts of linear transformations frequently in linear algebra courses and future courses. Linear algebra is viewed as essential for many students, and we need to equip them to be prepared to use these concepts in future courses and careers.

Acknowledgments

I thank my participants Perry, Summer, and Jed for their time, and my thesis advisor Dr. Gulden Karakok for her guidance.

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