Incorporating Generalization into University Classrooms: An Emerging Distinction from Instructors

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Abstract: In this paper we share findings from six interviews with instructors on whether, and how, they attend to generalization in their teaching. In particular, the interviews highlighted a distinction between generalizing in the classroom as being teacher or student generated. This study furthers our understanding of generalization beyond student activity and opens us to further questions such as how, if at all, students understand this distinction in the classroom.

Keywords: generalization, instructional practice, teaching

Introduction

Generalization is a foundational mathematical activity, and there is much agreement in the field that generalization is an important practice for students to learn (Davydov, 1972/1990, Kieran, 2007; Mason, Stacey, & Burton, 2010; Mata-Pereira & da Ponte, 2017). While studies have explored students' generalizing activity through interviews (e.g., Jones & Dorko, 2015) and small group teaching experiments (e.g., Reed & Lockwood (2021)), much less has been investigated about generalization occurring in classrooms. In particular, although generalization is clearly a fundamental and important mathematical practice, little is known about the extent to which generalization is explicitly or implicitly discussed, taught, or presented in university classrooms (if at all). The work presented in this paper is part of a larger study that seeks to explore both how generalization is currently presented in K-16 classrooms and how generalization might be effectively taught and incorporated into K-16 classrooms. We focus here on the postsecondary setting, examining ways in which university instructors may attend to and incorporate generalization into their teaching. In this paper, we share findings from interviews with six university instructors, in which they discussed the practice of generalization and ways in which they think about and incorporate generalization in their teaching practice. These interviews revealed important insights about the nature of generalization, highlighting different ways in which practicing instructors think about and treat generalization in their classrooms. Given that relatively little has been explored about the teaching of generalization, especially at the postsecondary level, our broad research goal is to engage in initial, exploratory work to investigate how generalization is actually presented and taught in postsecondary classrooms. With this broad goal in mind, in this paper, we seek to answer the following research questions:

1. To what extent do instructors incorporate generalization in their teaching at the university level? In particular, to what extent did the instructors we interviewed attend to generalizing in their classrooms?
2. What insights about the nature of generalization emerged from conversations with practicing postsecondary instructors?

Background Literature

Generalization is a fundamental mathematical practice, and researchers (Ellis, 2007; Lannin, 2005; Peirce, 1902; Rivera & Becker, 2007, 2008), curriculum designers (e.g., Hirsch et al., 2007; Lappan et al., 2006) and policymakers (Council of Chief State School Officers, 2010) emphasize generalization as important. However, even given its status as a valued mathematical practice, there is ample evidence that students face difficulties in engaging in generalizing; for example they are not always successful in stating and evaluating general statements or in recognizing and articulating patterns (Blanton & Kaput, 2002; Čadez & Kolar, 2015; English & Warren, 1995; Lannin, 2005; Mason, 1996; Moss, Beatty, McNab, & Eisenband, 2006; Orton & Orton, 1994; Rivera & Becker, 2008). Much work has been done in the field to help students develop a deeper understanding of generalization and to gain proficiency with engaging in generalizing activity. In some cases, such attempts have taken the form of conducting theoretical (e.g., Harel and Tall, 1991) and empirical (e.g., Ellis, 2007; Ellis et al., 2017) explorations of different ways of characterizing generalizing activity, or of designing domain-specific tasks to help students engage in rich generalizing activity (e.g., Reed & Lockwood (2021)).

Many of the studies that have explored students' generalizing activity have focused on students' generalizing in interview settings but not within classrooms themselves (e.g., Dorko & Jones, 2015; Ellis, 2007; Reed & Lockwood, 2021). Further, such work has concentrated on *students'* engagement and has not interrogated or investigated teachers' experiences with thinking about or promoting generalizations; neither has it explored occurrences of generalization within classrooms. A goal of this study, then, is to begin to examine instructors' perspective of generalization, particularly within the context of their teaching practice. Compared to the wealth of literature on students' generalizing activity, considerably fewer studies have explored generalization in teaching. Indeed, the relatively limited research on teachers' support of students' generalization suggests that it is challenging for teachers to fostering correct generalization among students (Callejo & Zapatera, 2017; Cockburn, 2012; Mouhayar & Jurdak, 2012). We thus see value in trying to better understand ways to support teachers in promoting generalization in their classrooms.

Theoretical Perspective – Characterizing Generalization

There are a number of ways in which generalization has been defined in the literature, and in this section, we clarify how we characterize generalization in this study. Generalization has been presented both from an individual, cognitive perspective (e.g., Carraher, Martinez, & Schliemann, 2008; Davydov, 1990; Harel & Tall, 1991; Kaput, 1999) and as a social construct that involves discourse and activity (e.g., Doerfler, 1991; Jurow, 2004; Latour, 1987). We broadly adopt this social perspective by framing generalization within contexts and interactions. We follow Ellis (2007) and Ellis et al. (2017) in characterizing generalization as involving any of the following actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated. and/or (c) deriving broader results from particular cases.

There are a number of ways in which researchers have distinguished between aspects of generalizing. For example, Harel and Tall (1991) differentiate between expansive, reconstructive, and disjunctive generalization, and Harel (2001) has offered distinctions between result pattern generalization and process pattern generalization. Ellis also created a generalization taxonomy to focus on different kinds of generalizing actions (2007). Others have also built upon Ellis' (2007) taxonomy to articulate different kinds of generalization actions that emerged for students in a variety of domains and age ranges, such as describing the *Relating-Forming-Extending (RFE) Framework* (Ellis et al., 2017), which distinguished three major categories of generalizing activity: Relating, forming, and extending. When we consider students' generalizing activity, we often view such activity in terms of these generalizing actions. We mention these constructs here to demonstrate the many ways in which researchers have attempted to characterize salient aspects of generalizing. Each of these distinctions and characterizations are designed to shed some light on a feature of generalization, be it ways in which students might engage in generalizing activity, or ways in which we as researchers can identify, understand, and interpret generalization activity. We seek to understand relevant distinctions among generalization that might inform our understanding of generalization in the undergraduate classroom in particular, and our work contributes to these existing distinctions in the literature.

Many of the perspectives shared to this point have focused on the generalizing actions of students. However, as we will see in this paper, the results from the interviews shed light on additional distinctions that we find noteworthy and that are related to classroom contexts. So, while we frame our work in existing literature and characterizations of generalization, our findings will reveal what we think are novel distinctions that our instructors attended to and viewed as relevant. These provide additional insight into how we might think about generalization broadly and how we might encourage students to engage in generalization.

Methods

Data Collection

We report on individual interviews conducted with university instructors. These interviews are the first phase in a larger project in which we aim to create a professional learning community among 3-4 instructors over the course of the academic year, engaging them in professional development about the nature and practice of generalization in their teaching. We sent recruitment emails to 15 instructors at a large public university, inviting them to participate in an interview. Six instructors responded, and we interviewed all of them. The participants represented a wide range of the department (information is found in Table 1). We conducted individual, ~90 minute interviews over Zoom with the participant and both authors of this paper.

Table 1. Pseudonym, position title, experience, and courses typically taught for each participant.

Pseudonym	Position Title	Years teaching	Courses typically taught
Participant A	Senior Instructor	9	calculus, differential equations, linear algebra
Participant B	Associate Professor	9	discrete math, advanced calculus
Participant C	Professor	32	upper-level geometry courses
Participant D	Senior Instructor	10	calculus, differential equations, discrete math, linear algebra
Participant E	Senior Instructor	9	calculus, differential equations, linear algebra
Participant F	Instructor	1	college algebra, pre-calculus, calculus

Our goal with these interviews was to get a sense of the instructors' approaches to teaching, to identify their initial views and definition of generalization, and to understand if and how they incorporated generalization into their teaching. To this end, we asked demographic questions, general questions about their teaching, their initial characterizations of generalization, and questions about teaching generalization, including whether and how they attended to generalization in their own classrooms. After they presented their own initial characterization, we provided them with our working characterization of generalization from Ellis (2007): "Generalization involves engaging in at least one of three activities: a) identifying commonality across cases, b) extending one's reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases" (p. 444).

Data Analysis

The interviews were transcribed. The first author wrote summaries of each interview to get a better understanding of each participant's views of generalization in their teaching by identifying each of the participants' initial characterizations of generalization and comparing/contrasting them against each other. Using MAXQDA, she engaged in an open coding of the interviews for all the examples of generalization provided in the interviews (the final categories were: examples from courses taught, participant's own mathematical experiences, research experiences, and hypothetical examples) and she then used the coded sections to create a table of examples of generalization that the participants described. To identify themes and the results in this paper, both authors had ongoing conversations about what was appearing in the data through the participants' characterizations and examples of generalization, and together we engaged in initial open coding based on the different ways in which the participants described generalization in their classrooms. Several key distinctions emerged from this initial open coding, and from this we articulated the results given below, drawing on examples from the data, verifying our initial coding by repeatedly reviewing the transcripts, and having ongoing conversations about the examples given by instructors.

Results

Overall, the interviews yielded new insights about postsecondary instructors' views of and incorporation of generalization in their classrooms. We found that these instructors had a rich repository of characterizations and examples of generalization that went beyond our current understanding of generalization. We saw a variety of ways in which teachers engaged, or did not engage, in generalizing activity in the classroom, and they varied in the extent to which they had thought about generalization prior to the interviews. Indeed, it is noteworthy that some instructors were in fact quite attentive to generalization in their teaching, and they shared thoughtful insights about generalizing in their teaching.

Due to space, we can only present one key finding from these interviews. We describe one main distinction about generalization in the postsecondary classroom that emerged from the interviews, and we share examples and discuss it. This distinction, and the participants' discussion about it, informs our understanding of generalization in postsecondary classrooms, highlighting a variety of ways in which generalization might be incorporated and taught.

An Emerging Distinction between Generalizations in the Classroom

In analyzing interview data, a key distinction emerged about generalization in the classroom as to whether a generalization is instructor or student generated. As noted, many of the ways in which previous research has characterized generalization has been focused on students' generalizing actions – that is, how we might foster student engagement in particular generalizing actions such as relating, forming, and extending. While we maintain that such a focus on students' generalization is important, in our conversations with instructors we gained insights into broader ways in which they were conceiving of generalization, and we identified additional ways in which generalization might actually arise in classroom settings. We acknowledge that our current understanding of this distinction is not comprehensive (and we discuss limitations below). However, this distinction sheds light on nuanced ways in which generalization might emerge in a postsecondary classroom, and we found it compelling and enriching to consider.

As the instructors described generalization in their classrooms, it became clear that there was variety in the extent to which students themselves had opportunities to engage in generalizing activity. This distinction was most clearly articulated by Participant C. The following exchange shows Participant C initially describing a distinction he saw between two different ways in which generalization might be presented – whether he or the students were doing the generalizing. This exchange occurred toward the beginning of his interview when he was reflecting on characterizations of generalization:

> *Participant C:* … the tension is between generalization as a pedagogical strategy, which is what I think about most of what I do in the classroom to bring students to the stage of understanding more general idea framework context by starting with simpler such things. And there is separately teaching students to generalize themselves. In one case I'm doing the generalizing –
> *Int. 1:* Yeah.
> *Participant C:* – in order to give students something, to anchor their understanding of the more complicated scenario too. In the other case, you're trying to get students to recognize some new situation as a generalization of some old situation. Those are, those are at least subtly different because in one case I'm doing the generalizing for them and the other case they're doing it.

This underscores the idea that there may be two separate objectives for generalization in the classroom – one is to help students think about and handle general ideas or concepts, and the other is to teach students to generalize themselves. Participant C was clearly thoughtful in his own consideration of these different objectives, and he went on to say that in much of his teaching he tends to do more of the former and less of the latter.

This distinction was borne out in other interviews, too (not as explicitly, but participants did share whether they or their students engaged in generalizing activity). Overall, two participants said that they attempted to give students opportunities to engage in generalizing, two said that they as instructors generalized or discussed generalization but their students themselves did not generalize, and two indicated that they did not bring up generalizing in their classes at all. We now provide three examples from different participants (one that is instructor generated, one that is student generated, and one in which both parties engage in generalization) to highlight this distinction between instructor generated and student generated generalizations.

Example 1: instructor generated. The following example with Participant B demonstrates an instance of the instructor, and not the students, doing the generalizing activity. In this example, Participant B described a conversation she had with students during lecture about the nature of equivalence classes; that conversation prompted her to model generalizing by showing how a broader statement can sometimes be made when we notice a common feature of particular examples. As Participant C described, she was making a general connection for her students.

> *Participant B:* … Um, but it's sort of interesting because I, one other thing that sort of popped in my head was something a little bit different, um, where you can today, like today in [discrete mathematics], I, I, I was talking about equivalence classes, and I, I wrote down an example of an equivalence relation and then I, I s- sort of picked out some elements from the set and said, "Let's write down the equivalence class for this guy, and the equivalence class for this guy, and I went through a couple of them. Um, and then I said, you know, "Notice that if two of the elements of the set are equivalent to one another, then they have the equivalence classes." Um, and then I said, **"Oh, it turns out that this is always true," and I wrote down the statement of the theorem** and I, I sort of s- we didn't really manage to get through the proof, but we started talking about the proof a little bit**. Um, so like I mean, I guess that in some sense is generalization, right, because you're starting with sort of a specific example where a certain thing is true, but then you're immediately jumping to, obviously, this broad class of equivalence relations**. Um, so, so, in that case, the generalization kind of happened in an instant.

Example 2: student generated. As an example of student generated generalization, we share a case where Participant D described intentionally giving students chances to generalize. In this example, he discussed scaffolding tasks in which students might relate (in the sense of Ellis et al.'s (2017) framework) other proofs of irrationality to their initial experience with the square root of 2, ultimately extending ideas to more general cases. Here, we interpret that the instructor was attempting, through a choice of tasks, to provide opportunities for students to engage in generalizing activity and to formulate generalizations for themselves.

> *Participant D:* So for, for another example, in my eCampus course, um, I have the students do discussion board posts with each other where they're kind of required to work with each other on some exercises. And I think like in one of my videos, I, I go through the classic proof that the square root of two is irrational by contradiction. And some of the exercises I'll put on the activity are like, okay, can you prove the square root of three is irrational? Can you prove the square root of five is rational? **Can you, can you generalize this to proving other things are irrational?** This is as a, as something where, okay, I've, I've done the exercise for square root of two. Can you now see that this, this, this, these ideas generally apply to other, other proofs of irrationality of square roots?

Example 3: instructor and student generated. Finally, we offer an example in which both instructors and students together contribute to a generalization in the classroom. Participant A described his idea of "ungeneralizing" a math concept to a more concrete scenario that students could then use to help rebuild the more general concept themselves. In this case, we see the instructor's generalization served to support and direct student generated generalizations. In particular, he worked to find relevant examples and analogies, with the hope that students could then generalize from them. He also commented that up until the interviews with us, he had been thinking of these as analogies as opposed to instances of generalization, but during the interview he realized that generalization more accurately described his teaching intentions.

> *Participant A:* Well, it's, i-it, this is like hard 'cause I feel like what my, **part of my role as the teacher is, is, is to, is to take the abstract idea and provide specific examples to, that I choose carefully so that the students can then generalize from them**, to like to gain that deeper, intuitive understanding of the abstract concept through my ungeneralizing it.
> *Int. 1:* Mmm. So like, you're doing the ungeneralizing for them, is what you're saying?
> *Participant A:* So I think about – so that they can regeneralize themselves.
> *Int. 1:* That's interesting.
> *Participant A:* So thinking about like some multivariable calculus, I think is a great subject for this conversation because there's almost no new math to learn. There's just those concepts you've already known, but now applied in 3D.
> *Int. 1:* Mmm.
> *Participant A:* And so, so much of the class is like, remember how it used to work? Now it works this way and here are the differences. And so there's great ways to, (laughs) at least I think there's great ways of ungeneralizing the idea of what is a parameter in a vector valued function, and how does, what does it mean for a parameter to represent arc length versus represent time. And ungeneralizing that into different ways that different types of vehicles keep track of how, of their age. So if you look at like a car like you and I own, it has an odometer and its age is how many miles it's driven. But if you go to a construction site and you look at like an excavator, it doesn't have an odometer, it has an hours. How many hours has this machine been operated? And so there's these anti-generalizations, these analogies that I try to use so that the students can understand the abstract concepts so that they can begin to generalize.

While these are just brief examples of different ways that generalization might arise in postsecondary classrooms, we think that they highlight potential ways in which students may gain access to ideas related to generalization, even if instructors are making or describing generalizations. This distinction highlights that generalization might arise in a variety of ways – it does not only have to involve students themselves generalizing, but it might also involve instructors making clear attempts to explain general concepts, make connections, or generalize ideas for students. This is particularly noteworthy for some of the specific mathematical content that these instructors teach, which involves abstract concepts.

We want to emphasize that when instructors (and not students) generalized, this did not always suggest that instructors did not think it was worthwhile for students to generalize. Rather, they attempted to do some of the work with generalizing for interesting and nuanced reasons. Participant A's use of analogies involved him making connections, but it was for the purpose of setting up students to be able to engage in generalizing activity. As another example, Participant C gave an example of wanting their students to not overgeneralize in geometry, and he explained that focusing on 3 dimensions was a worthwhile use of their class time (and not, say, 17 dimensions, which would be a valid generalization but not one he wanted students to explore given the scope of his class). Thus, while he was supportive of the idea of generalizing and saw its value for students, in teaching that course, he was more inclined to direct students' generalizing so they would not get too far afield in their explorations.

As a final note, two of the instructors had not thought about whether or not they or their students had opportunities to generalize in class, but they were reflective about it once we raised the idea in the interviews. For example, Instructor F was able to reimagine the way she taught optimization problems to her calculus students as engaging in generalization of a procedure from a set of examples, and she noted that thinking of this activity in terms of generalization gave her new insight into how to explain optimization to her students.

Discussion and Limitations

We believe our initial instructor interviews raise questions for subsequent research about the effectiveness of student versus instructor generated generalizations in the classroom. That is, while it is clearly desirable to have students engage with generalizing activity, this distinction suggests opportunity for explorations into what (if anything) students gain from instructor's attention to generalizing. Based on prior work about students' experiences in lectures (e.g., Lew et al., 2016), it is likely that students may not necessarily pick up on points that instructors are trying to make about generalization. However, it is worth investigating whether there might be any benefit to instructors making explicit comments about generalization in their postsecondary classrooms, particularly as students encounter advanced, abstract concepts. Further, from a professional development perspective, it is worth investigating productive ways to engage instructors with exploring ways to incorporate generalization in their classrooms.

One limitation of this work is our reliance on instructor reports on their teaching, rather than observing their teaching (this is due in part to the pandemic and remote instruction); it would be good to observe teachers to see how generalization occurs in their classes. However, as we have seen, we contend that their self-reported data still offers useful insights in thinking about the nature of generalization and how it might be incorporated into postsecondary classrooms. We also acknowledge that we only interviewed six instructors, and more distinctions and nuances may arise by interviewing or observing more university instructors.

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