Student Mathematical Reasoning and Construction in a Digital Environment with Regard to Conceptions of Circles

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This study addresses what students' conceptions of circles are and how students' interaction with a digital environment influences their development of those conceptions. Two undergraduates from calculus 1 courses participated in exploratory teaching interviews for two one-hour sessions in fall 2019. In this study, we focus on moments in which the students' mathematical reasoning in regard to circles was on display. The two students demonstrated similar conceptions of and construction of circles on a physical medium at the start of Day 1. However, they interacted differently with the digital environment in the rest of the teaching interviews.. Consequently, their conceptions at the end of Day 2 were markedly different. We will discuss the progression of each student and how their interactions with the digital environment influenced their conceptions of circles, paying particular attention to the students' reasoning throughout.

Keywords: Geometry, digital environment, Euclidean construction, circle conception, teaching geometry

Visualization plays an important role in the teaching and learning of geometry as geometry pertains to the study of shapes and figures. On the other hand, research has reported that many students struggle with visualizing geometric shapes and figures that properly convey specific mathematical concepts (Walcott et al., 2009). As guiding students from a general conception (e.g. "This is an equilateral triangle because it looks like one") to a very specific conception of shapes (e.g. "This is an equilateral triangle because all sides of the triangle are equal") is a key element of geometry education (Jones, 2000), digital software and environments are considered devices to combat students' conceptions of geometric shapes and challenge students to rely on specific properties of shapes instead (Walcot et al., 2009). In particular, mathematics education research has shown that the use of digital manipulatives developed from dynamic geometry software such as GeoGebra could be helpful to improve their visualization skill (Baki et al, 2011; Dogan & Içel, 2011; Laborde, 2006). In line with this standpoint,this study focuses on students' conceptions of circles and how digital environments may influence students' development of conceptions of circles. To be specific, we developed GeoGebra applets and implemented them to see how digital environments may influence students' development and conception of circles. By analyzing two undergraduate students' construction and use of circles in the GeoGebra applets during two days of exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) we address the following research questions:

1. What are students' conceptions of circles as they are evoked in geometry construction problems?
2. How are students' conceptions of circles evoked as they interact with the digital environment?

Theoretical Framework

When we analyzed students' conception of circles in the digital environment, we took for granted that different students may conceive of circles in different ways. Indeed, we noticed that students mainly used two conceptions. In some moments, students' reasoning and construction were based on circles as a geometric shape and in other moments, their reasoning and construction were based on circles as a geometric object (Sakauye, 2020).

Geometric shape. Students who think of circles as a geometric shape do not appear to consider the properties of circles in their reasonings or constructions. That is, they describe and construct circles simply as a shape and do not take into account any properties of the circle in doing so. We can categorize students' conceptions of circles as a *geometric shape* if their explanations and constructions do not include the use of the properties of a circle.

Geometric object. Students who think of circles as a geometric object include the properties of circles in their reasonings and constructions. We can categorize students' conceptions of circles as a *geometric object* if their explanations and constructions include the use of the properties of a circle, such as that all points on the circle are a fixed length from the center.

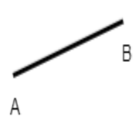
Our categorization of students' conceptions of circles is in line with the van Hiele levels of geometric understanding (Mason, 2009). In particular, the conception of circles as a geometric shape corresponds to van Hiele level 1 because students at van Hiele level 1 recognise shapes by their appearance, not by their properties (Mason, 2009). The conception of circles as a geometric object also corresponds to van Hiele level 2 or higher since students at that level recognise shapes by their properties (Mason, 2009).

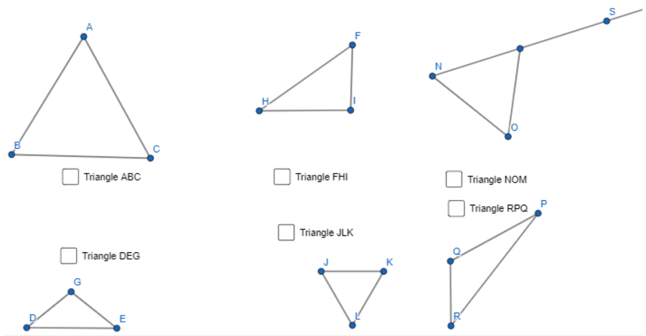
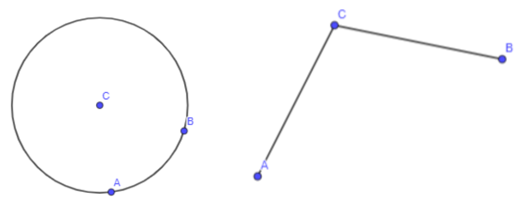
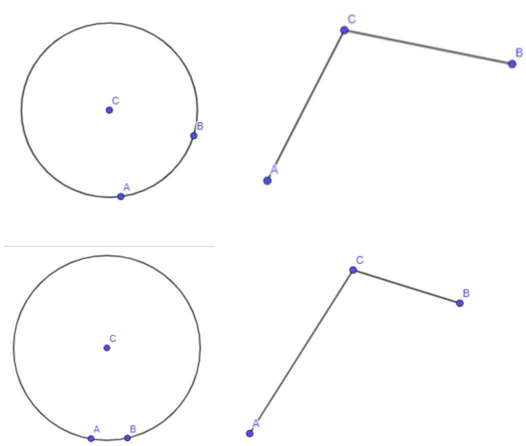
Methodology

Two undergraduates from calculus 1 courses individually and voluntarily participated in exploratory teaching interviews (ETIs) (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) for two days in fall 2019. We selected students from calculus 1 classes because they were already introduced to geometry concepts like circles and equilateral triangles, but would not have learned the formal constructions yet - if they had already learned the constructions, their responses in this study would be reports of what they learned in class, not authentic reactions to the digital environment. The first author of this paper acted as the teacher-researcher, while the second author and another researcher served as witnesses of the ETIs. Data included videotapes of sessions, audio recordings, copies of students' written work, as well as screen recordings on a tablet used to access the digital software (a GeoGebra web page designed by Fischer (2019)). In this study we focus on moments in which the students' reasoning in regard to circles was on display, both in the digital software and on a physical medium.

Structure of the ETIs

The experiment took place over two days. A list of the tasks and the accompanying diagrams are included in the table below.

Task name	Diagram	Task instructions
Pretest Day 1		Given a segment AB, construct an equilateral triangle using a ruler and compass on paper.

True or false statement Day 1	☐ A line is a one-dimensional figure with no thickness that extends indefinitely. ☐ A line segment is a bounded or finite line ☐ A circle is a set of points in a plane, equidistant from a center point. ☐ A triangle is a three-sided polygon.	Answer whether the question about circles is true or false.
Triangle exercise Day 1		Check if each triangle is an equilateral triangle using the circle tool.
Circle exercise (original) Day 1		For the diagram with the curve of the circle drawn: Are the points on the circle equidistant from the center of the circle? For the diagram without the curve of the circle drawn: Are the two segments equidistant?
Circle exercise (revised) Day 2		For each of the diagrams with the curve of the circle drawn: Are the points on the circle equidistant from the center of the circle? For each of the diagrams without the curve of the circle drawn: Are the two segments equidistant?
Construction of equilateral triangle in GeoGebra Day 2		Given a segment, construct an equilateral triangle using the circle tool and the segment tool in GeoGebra.

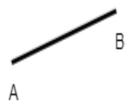
Posttest Day 2		Given a segment, construct an equilateral triangle using a ruler and compass on paper.

Table 1. A list of the tasks used in this study, including the instructions given. The original circle exercise can be accessed here https://www.geogebra.org/m/kmwyaua8 *and the rest of the tasks in GeoGebra can be accessed here* https://www.geogebra.org/m/hfqxjr4k

Data Analysis

The data collected was part of a larger study. In this paper, we report only our analysis related to student conceptions of circles. Drawing from recordings of the ETIs, we analyzed the students' words, gestures, and diagrams they produced by moments, to capture changes in students' reasoning.

Ongoing analysis was done spontaneously in the ETIs, described by Steffe and Thompson (2000, p. 276) as "creat[ing] new hypotheses and situations on-the-spot" in exploratory teaching. Thus, we formed hypotheses about the students' reasoning, and tested those hypotheses immediately in the form of questions. Our ongoing analysis also informed our decision to include additional exercises for Day 2, as described in the methodology.

After the ETIs were completed, we used Corbin and Strauss's (2014) grounded theory in the sense that two categories of students' conception of circles emerged from the data analysis: circle as a geometric shape and circle as a geometric object (see theoretical framework). We relied on these two categories to analyze our data and arrive at our results. Then, we examined what the students' conception of circles were and how they were evoked in each stage of the ETIs. Finally, we compared the two students in terms of their conception of circles and how their interactions differed in the digital environment.

Results

We found that the two students demonstrated similar conceptions of circles in the pretest as they constructed circles in the physical medium. During Day 1, they also displayed similar conceptions of circles in the digital environment. However, on Day 2, they displayed different trajectories of conceptual development in regard to circles. In this results section, we report our analysis on the different progression of each student and how their interactions with the digital environment influenced the development of their conceptions of circles differently.

Pretest Results

The main task in the pretest was to construct an equilateral triangle using a ruler and compass. A typical way to construct an equilateral triangle is constructing the intersection between two circles, each centered at one endpoint of the segment and with the segment as the radius. Neither student constructed the triangle in this way.

When constructing an equilateral triangle during the pretest, both students first checked the tick marks on the ruler to measure the length of the given segment, then attempted to construct two other sides of a triangle from each endpoint of the given segment. Both also checked the length of each segment with the use of a ruler as a measuring tool. Athena explained that what she had constructed was an equilateral triangle because all of the sides were "the same length AB", which was the length of the given segment she had measured. Faith had similar reasoning, that she only needed to measure segment AB in order to replicate the same measured length on the sides of the triangle. Since they both relied on measurement as their method of construction, neither seemed to consider the compass (or circles) as a necessary tool for construction of an equilateral triangle, and thus did not use it.

Day 1 Results

At the beginning of Day 1, the interviewer asked the students if a circle was a set of points on a plane equidistant from a center point. Both students verbally accepted the statement as presented to them in the digital environment. While completing the triangle exercises, both students had a specific methodology for constructing circles (shown in Figure 1). They created a point, T, in the middle of the triangle, intended to be the center of triangle ABC. Then, they dragged their pens back and forth from the point T, (which enlarges and shrinks the radius of the electronically drawn circle). When it appeared the shape of the circle included points A, B, and C, they lifted their pens off the screen, which constructed the circle centered at T. When they lifted their pens, they had not chosen a radius, so the pens were not directed to any point in particular, and the GeoGebra software automatically created a point where the pens were lifted off the screen. They both used a similar method to construct circles in the circle exercises.

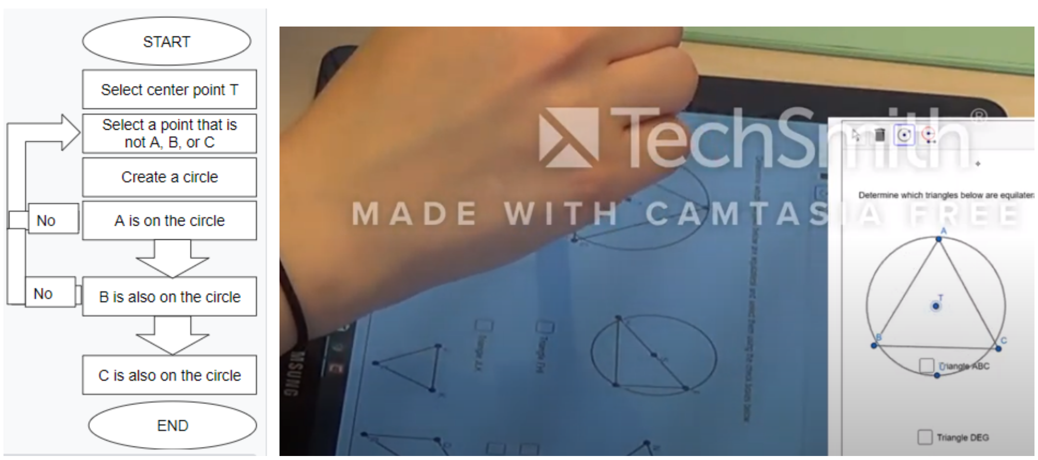


Figure 1. A flowchart of the method both Athena and Faith used to construct circles on Day 1 (left) and a screenshot of Athena using the method to construct a circle centered at T (right).

Once the circles were constructed, both students included the property that all points on a circle are equidistant from the center in their reasoning.

Findings from Day 1. The students' conceptions of circles remained similar on Day 1. Athena and Faith's verbal explanations demonstrated their conception of circles as geometric objects; however, they did not consider the properties of a circle in their constructions. Neither used the property that radii are equal in length to construct a circle, because neither selected a point on the circle as the endpoint of a radius. They instead dragged their pens across the screen, which enlarged the electronically drawn circle, so they could determine which points "look like" they are on the circle. Both students viewed construction as drawing the geometric shape based on visual stimulus (GeoGebra) because neither Athena nor Faith included radii in their construction of circles. Since they consider the properties of a circle (including equidistant radii) in their reasoning, but not in their construction of circles, we categorize their reasoning about circles as conceptions of geometric objects and their conceptions of circle construction as conceptions of geometric shapes.

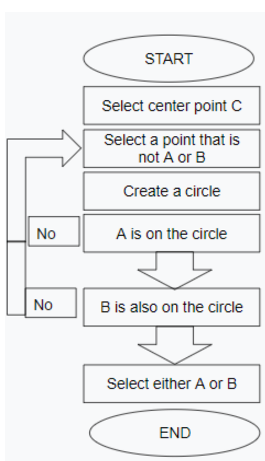
Results of Day 2

In response to the students' consistent strategies in constructing circles based on the visual shape of the circle demonstrated in Day 1, the research team developed and implemented a new requirement for construction in GeoGebra: the instruction "don't create new points". As GeoGebra will construct a point if the point selected by the user was not previously defined, this seemed like an effective intervention for the students' reliance on visual stimulus in construction. In this section, we describe individually each student's response to the new requirement in regard to her reasoning and construction of circles and analyze the major differences between the two students' conceptions on Day 2.

Findings from Day 2. We found that during Day 2, two students' construction of circles changed differently in regard to their integration of the new requirement.

Athena's reasoning was similar to her explanations during Day 1. Although the interviewer intervened with a new requirement for GeoGebra use, such instructional intervention did not have an impact on her conception of circles. Rather, her verbal reasoning continued to demonstrate a conception of circles as geometric objects. In regard to construction of circles, Athena did not seem to change her conception of construction of circles as geometric shapes. The change in her method (i.e. tracing a circle before selecting a point on the outside of the circle) seems to indicate that her accommodation of the intervention was to add a step to her previous method (checking that the point she chose would not create a new point). Instead of changing her conception of circles to a geometric object (e.g. including the property that a circle has a radius), she continued to drag her pen back and forth, relying on the shape of the constructed circle to determine the size of the circle.

On the other hand, Faith's verbal reasoning and construction of circles demonstrated a conception of circles as geometric objects by the end of Day 2. Following the new requirement, to not construct another point when constructing a circle in GeoGebra, she selected a point as the center of the circle, then dragged her pen across a segment she had chosen as a radius to select the endpoint of the radius on the circle. As she had chosen a radius in the construction of circles, we considered her conception of circles as geometric objects.



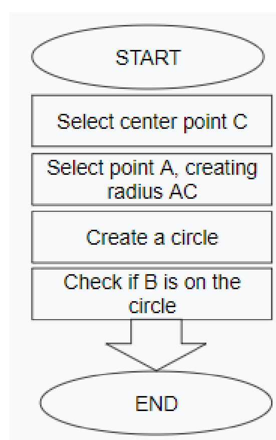


Figure 2. A flowchart of the method Athena (left) and Faith (right) used to construct circles in Day 2.

Using the digital environment to adapt their reasoning, the students fulfilled the new construction requirement. Athena simply added to her method of constructing circles - her method is different only in that instead of creating a new point on blank space, she traced a curve from the blank space to a previously defined point. The way she constructed circles in the digital environment did not include her reasoning about the properties of circles. Faith, on the other hand, seemed to change her method in order to integrate the new requirement. Instead of relying on the visual shape of the circle in construction, Faith used the properties of a circle (that all the radii were equal in length) to construct her circles.

Posttest Results

The research team administered the posttest on paper in order to observe what impact, if any, the transition back to physical materials would have on the students' reasoning. We observed that students' reasoning in the posttest had changed significantly from the pretest.

Findings from the Posttest. Both students' conceptions of circles in construction and reasoning did not change in the posttest. Athena's verbal reasoning about the properties of a circle demonstrated her conception of circles as geometric objects; however, her use of the compass without selecting radii of the circle in her construction demonstrates her conception of construction of circles as geometric shapes. Faith, on the other hand, selected radii in her construction of circles, and also reasoned using the property that radii of a circle are equal, demonstrating her conception of circles as geometric objects both in construction and verbal reasoning.

Conclusion

The GeoGebra environment created by Fischer (2019) was helpful to the students in this study as a concrete method of visualization for both their reasoning about and of circles. Athena commented on how interesting GeoGebra is and Faith said that GeoGebra "seemed like a cool program". In addition to the students' positive feedback about the application itself, GeoGebra seemed to help the students progress in their construction of an equilateral triangle and influenced their conception of circles.

Although both Athena and Faith began the ETIs with similar conceptions (demonstrating conceptions of circles as geometric objects verbally, but constructing circles with a conception of circles as geometric shapes), their conceptions at the end of the ETI were different. While Athena accommodated the teaching intervention without changing her conceptions of circle construction as a geometric shape, Faith incorporated the conception of circles as geometric objects in her constructions, with the inherent property that all radii of a circle are equal.

In regard to how the students' conceptions of circles were evoked as they interacted with the digital environment, our results indicate that the students demonstrated different conceptions of circles in their reasoning and construction. While further study is needed, the results of the students' changes in reasoning and construction suggest that GeoGebra has the potential to influence students' conception of geometric objects, such as circles, which would make it an invaluable tool in geometry classrooms, both in-person and online.

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