Mathematical Sameness: An Ill-Defined but Important Concept

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*Sameness is a notion that pervades mathematics through concepts like equality and isomorphism, but limited research has examined how mathematicians perceive sameness in mathematics. This study examines survey responses from mathematicians on the nature of mathematical sameness. Themes highlighted from responses include philosophical, proof-based, and informal notions of mathematical sameness.*

*Keywords:* Isomorphism, Homomorphism, Advanced Mathematics, Sameness

## Background and Literature Review

Math educators have been studying students' understanding of equality for many years (e.g., Renwick, 1932; Kieran, 1981). Research on students' understandings of higher-level notions of sameness, such as isomorphism, began more recently (e.g., Dubinsky, Dautermann, Leron, & Zazkis, 1994) and are still being studied (e.g., Melhuish, 2018; Rupnow, 2017). Nevertheless, research on types sameness, their utility, and problems for students' thinking have only recently been examined (Melhuish & Czocher, 2020). In this paper, we solicit experts' perspectives as we address the research question: How do algebraists describe mathematical sameness?

Prior research on equality, isomorphism, and homomorphism have largely focused on students' conceptions. Extensive research on students' understanding of equality in numerical contexts has connected students' understanding of the meaning of the equal sign to their later ability to understand and manipulate algebraic equations (e.g., Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Kieran, 1981). Research on isomorphism has examined how students determine if groups are isomorphic (e.g., Dubinsky et al., 1994; Leron, Hazzan, & Zazkis, 1995; Rupnow, 2017) as well as the use of properties of functions to draw conclusions about isomorphism and homomorphism (e.g., Melhuish, Lew, Hicks, & Kandasamy, 2020).

However, limited research has examined experts' conceptions of types of sameness. Weber and Alcock (2004) noted mathematicians' descriptions of group isomorphism as "re-labelings" or meaning groups were "essentially the same". Rupnow (2021) expanded on these notions with other instructor metaphors for isomorphism or homomorphism, such as "matching" and "structure-preservation". However, these descriptions related only to isomorphism and homomorphism, not to general descriptions of sameness in mathematics.

Cross-concept examinations of sameness have barely been examined. Melhuish and Czocher (2020) noted the context-dependent nature of sameness in mathematics that is not always fully elaborated when students are asked to respond to prompts related to sameness. For example, though $Z_3$ is isomorphic to a subgroup of $S_3$, we may not want students to suggest that $Z_3$ is a subgroup of $S_3$ because we want to attend to the elements or operations defined in the groups, not just structural similarity. However, the concepts that mathematicians think should be aligned has not been examined nor the ways in which they elaborate on the nature of sameness.

## Conceptual Framework

Our theoretical lens for analysis is conceptual metaphors (e.g., Lakoff & Núñez, 1997), in which a source domain is used to structure understanding of a target domain. For example, "Happiness is an equation" and "Happiness is a warm summer's day" are metaphors that might be used to describe the target domain (happiness) in terms of two different source domains (an equation and a warm summer's day). The two metaphors provide different ways of understanding happiness: happiness as a concept that can be broken into constituent pieces and combined in standard ways, or happiness as a concept that is focused in-the-moment.

Conceptual metaphors have previously been used in math education to examine functions in high school and linear algebra (Zandieh, Ellis, & Rasmussen, 2017) as well as to examine isomorphism and homomorphism in abstract algebra (Rupnow, 2021). Here we examine the nature of sameness (target domain) through a variety of metaphors addressing philosophical notions of sameness, proof-based notions of sameness, and informal notions of sameness. For example, "Sameness is a context-dependent concept" is a philosophical description of sameness that emphasizes sameness has many interpretations.

## Methods

Data were collected from a survey sent to every 4-year college or university math department that offers abstract algebra in the United States. This survey addressed how algebraists think about sameness in general and in specific mathematical contexts. Participants were 197 mathematicians from 173 institutions who had taught at least one abstract algebra or category theory course in the last five years. For this paper, only the first three questions from the survey, which broadly regard the nature of sameness in math, are discussed:

1. What does it mean to be the same in a math context? (Q1)
2. How do you know two things are the same in abstract algebra? (Q2)
3. How is "sameness" in abstract algebra similar or different from "sameness" in other branches of math? (Q3)

Responses to each question were coded independently, but each response could receive multiple codes. Descriptive coding (Saldaña, 2016) was used to formulate conceptual metaphor-based codes for use in thematic analysis (Braun & Clarke, 2006). In keeping with this approach, we used multiple rounds of coding (Anfara, Brown, & Mangione, 2002). Both authors coded independently, then came to consensus for each participant's response. Next, codes were organized according to theme. In this process, we concluded some codes should be split or combined. We then revised codes, regrouped codes, and recoded responses as necessary. Finally, we discussed revised codes to ensure consensus and consistency.

## Results

We highlight three categories of response that we observed in our coding: philosophical, proof-focused, and informal notions of sameness. We characterize types of responses in each of these broad categories. Frequencies of codes are provided in Table 1.

### Philosophical Notions of Sameness

In this section, we highlight participants' descriptions of sameness that relate to philosophical notions about the nature of sameness: sameness as context-dependent and sameness as context-bridging. Because all but 5 participants (97%) referred to sameness as context-dependent and 67% highlighted context-bridging, we showcase multiple responses characterizing these stances.

**Context-dependence.** We highlight five main flavors of context-dependent answer. These include general context-dependent responses, a focus on specific concepts that transcend disciplines, comparisons of sameness in different disciplines, and a focus on different strengths of sameness. Some responses combine these ideas.

A number of respondents wrote generally about sameness being context-dependent. A few believed that mathematical sameness was too vague of a concept to engage with in the survey. Many of these responses were also given the code *vague/don't engage*. For example, Participant 20 wrote, "I don't know because it's entirely unclear what "same" means here as you are using it in this survey or what it means in abstract algebra." Others elaborated; for instance: "First you need to decide which kind of "the same" you mean. Typically there is a list of features that need to be compared, either directly or by constructing the appropriate function" (Participant 92). Both of these responses highlight the variety of meanings that could be associated with sameness and emphasize the importance of ascertaining the correct type of "sameness" to respond.

*Table 1. Code Counts for Each Question and Across Questions*

| Group | Codes | Q1 n(%) | Q2 n(%) | Q3 n(%) | Distinct n(%) |
|---|---|---|---|---|---|
| | Vague/don't engage | 12(6%) | 5(3%) | 4(2%) | 16(8%) |
| Philosophical | Context-dependent | 107(54%) | 47(24%) | 146(74%) | 192(97%) |
| | Context-bridging | 19(10%) | 3(2%) | 123(62%) | 131(67%) |
| Proof Based | Bijective | 17(9%) | 30(15%) | 18(9%) | 50(25%) |
| | Cardinality | 3(2%) | 9(5%) | 6(3%) | 16(8%) |
| | Same properties | 8(4%) | 3(12%) | 10(5%) | 19(10%) |
| | Logically equivalent | 19(10%) | 30(15%) | 10(5%) | 46(23%) |
| | Computation | 4(2%) | 28(14%) | 7(4%) | 37(19%) |
| | Invertibility | 2(1%) | 7(4%) | 1(1%) | 9(5%) |
| | Mutual subsets | 2(1%) | 2(1%) | 1(1%) | 5(3%) |
| Informal | Same behavior | 52(26%) | 24(12%) | 29(15%) | 77(39%) |
| | Indistinguishable | 8(4%) | 1(1%) | 3(2%) | 11(6%) |
| | Mutual replaceability | 2(1%) | 1(1%) | 0(0%) | 2(1%) |
| | Equivalence | 22(11%) | 5(3%) | 8(4%) | 28(14%) |
| | Structure-preservation | 18(9%) | 29(15%) | 22(11%) | 50(25%) |
| | Operation-preservation | 1(1%) | 15(8%) | 8(4%) | 19(10%) |
| | Matching | 5(3%) | 15(8%) | 8(4%) | 21(11%) |
| | Relabeling | 16(8%) | 13(7%) | 6(3%) | 27(14%) |

Other participants highlighted differences between general terms used for sameness in math. For example, Participant 10 attended to which differences were important: "Same can mean many things—equal, isomorphic, equivalent. It's important to know what differences matter and what don't to determine what 'same' means." Here they listed three concepts that relate to sameness, while noting that "what differences matter" is central to knowing sameness.

Many participants highlighted specific types of sameness in different disciplines to illustrate context-dependence. For example, when responding to Q1, Participant 149 wrote:

> Many branches of mathematics concern themselves with classifying objects up to some notion of sameness. These notions vary by mathematical context. Topology, for example uses such notions as homeomorphic, homotopy equivalent and the like. Abstract algebra generally uses isomorphic….In general, when the objects are "sets with structure," sameness means that the sets map bijectively and the structures are respected by this mapping.

A key aspect here is examples of defined concepts to show that the contextual meaning of sameness is given by well-defined notions in various disciplines.

A number of participants focused on distinguishing between stronger and weaker types of sameness. Participants did this in two main ways: by discussing the strength of one or more notions of sameness from a particular part of mathematics or by contrasting the relative strength of notions of sameness in different disciplines. For example, Participant 15 noted:

> The main difference lies in what properties we want the sameness to capture, and whether we want a strong or weak notion. In category theory, for example, a weak notion of "sameness" (categorical equivalence, or isomorphism of skeletons) turns out to be much more fruitful/useful than the stronger version requiring a complete correspondence between them.

In addition to discussing the relative strength of two notions of sameness within the discipline of category theory, this participant is also discussing the utility of different strengths of sameness for gaining insight. An example contrasting strengths in different disciplines comes from Participant 42's response to Q3: "Compared to more classical subjects, isomorphism is similar to "congruence" in classical geometry but more flexible than "equality" in classical algebra." This participant seems to imply that equality is a stronger notion of sameness than isomorphism and congruence, but that isomorphism and congruence signify similar levels of sameness.

Other participants combined multiple aspects of context-dependence. For example:

> As in my earlier answers, "sameness" in abstract algebra is often taken to be the existence of an isomorphism. Sometimes, and especially in category theory, it is taken as a canonical isomorphism given by a universal property. Depending on the particular category used in another subject (e.g., homotopies in topology), we might get what appears to be a more flexible notion of sameness. In set theory, equality gives a much more rigid notion of sameness based on equality: having precisely the same elements. Having a bijection is typically not considered enough for sameness. Similarly, in geometry sameness is often treated more rigidly: points may look alike but are not the same. In computational problems, such as in public key cryptography, one only has sameness if the isomorphism can actually be computed in acceptable time. (Participant 14)

Here, the respondent listed six types of sameness and compared their relative strengths.

**Context-bridging.** Most explicit references to sameness as a context-bridging concept occurred in response to Q3, which prompted participants to compare and contrast sameness in abstract algebra with other math sameness. Some of these responses were brief, such as Participant 7's response: "Not different at all," suggesting that sameness has a universal meaning of some sort. Others elaborated on their view of sameness: "At the undergraduate level, I think it's not that much different from other classes like topology. Sameness usually means preservation of structure, and what abstract algebra is studying [is] structure" (Participant 3). Here the participant focused on structure-preservation as a cross-disciplinary idea.

A number of other responses, especially for the first question on sameness in math, connected to category theory: "If you have two objects in a category, they are the same if there is an isomorphism between the objects" (Participant 22). Although one might say that category theory is its own subject, we viewed this type of reference to category theory as a way to view math holistically. Thus, abstract references to objects being linked by an isomorphism within a 1-category were viewed as a way to view sameness in math as a unified concept.

**Both.** Many participants invoked ideas that combined context-bridging and context-dependent perspectives. Some invoked category theory to provide a framework for comparing sameness in different disciplines, such as Participant 21:

> If we're talking about isomorphism in a given category, then the similarity is through the definition of isomorphism, and the differences will be encoded in the properties of the category. Algebraic isomorphisms must preserve algebraic structures, while this is not a valid notion for homeomorphisms of space….

The participant starts by writing about sameness as a consistent concept interpretable across various places in mathematics via category theory, which implies a context-bridging aspect of sameness, but they also focus on sameness notions in different disciplines, highlighting their differences and therefore the contextual dependence on discipline for these notions of sameness.

Others wove context-bridging and context-dependent notions together through highlighting the specific shared and distinct aspects of types of sameness.

> As noted above, this notion of structural equivalence appears in many areas of mathematics. Isomorphism in abstract algebra is, for instance, analogous to homeomorphism in topology, similarity in geometry, equivalent statements in formal logic. It is distinct from these it that it is defined based on how binary structures behave under their operations.

Here, Participant 132 equates sameness with "structural equivalence", which is a topic relevant across mathematics. However, they acknowledge that the specific objects differ in each discipline (e.g., binary structures with operations in algebra).

**Proof-focused Notions of Sameness**

Some participants used language associated with uniqueness proofs or sameness of object including same properties, establishing a bijection or same cardinality, deduction of logically equivalent statements, verification via computation, finding inverse mapping pairs, and finding mutual subsets. Although these codes were rare (each in under 15% of responses to each question), as a group they provide another way of viewing mathematical sameness.

We first address the single property metaphors: bijection, cardinality, and same properties. These relate to determining a specific well-defined characteristic is shared by two objects: numerosity (for bijection and cardinality) or other properties like being abelian (for same properties). For example, finding a bijection was a common aspect of describing isomorphism in abstract algebra for question two: "For most purposes in abstract algebra, two objects are the same if they are isomorphic, i.e., if there exists a structure-preserving bijection between the two objects" (Participant 46). These notions of sameness generally relate to a given definition of sameness (e.g., isomorphism) or can apply to simple contexts like sets to provide something to consider when determining sameness of objects. Lack of alignment in such cases would be useful in a proof context to verify objects are not the same.

The broad verification metaphors observed include deducing logically equivalent statements and verification via computation. These can be viewed as two ways of proving equivalence of objects: being able to use theorems and logic to show uniqueness or equivalence and exhibiting a mapping or invariant that satisfies a definition. For example, consider Participant 21, who highlighted both logical equivalence and verification by computing an invariant:

> …to be exactly the same, they would have to be identical, which would require proving something that equates the definitions of the two objects.…Or alternatively, you can invoke classification theorems, for example all groups with two elements are isomorphic. Or you may be able to classify an object up to isomorphism by computing its invariants, for example by finding the dimension of a vector space.

Notice, logical equivalence appears through "equating the definitions of the two objects" and "classification theorems" while computation appears through computing invariants. Both techniques can provide full proofs of sameness.

Two narrow verification metaphors also arose: finding inverse pairs and finding mutual subsets. While not prevalent in the data (5% and 3% of respondents), these metaphors can be viewed as specific techniques for demonstrating equivalence. For example, Participant 19 noted, "To prove two sets are the same, then we show that each is a subset of the other," which can be viewed as a technique for demonstrating sameness of sets. Similarly, for isomorphism, mutual inverses were described as a technique for demonstrating sameness of structures: "By constructing the two morphisms that are inverses. Widely used constructions are formulated as theorems, like the First Isomorphism Theorem" (Participant 108). While these techniques can lead to proofs of sameness of objects, they only apply in certain disciplines. Mutual inverses are only sensible in a function context whereas mutual subsets are only sensible in a set context.

**Informal Notions of Sameness**

Participants used a number of informal terms to highlight sameness: same behavior, indistinguishable, mutual replaceability, equivalence, operation-preservation, structure-preservation, relabeling, and matching. These notions appeared in over 40% of responses.

Same behavior was the most common of the informal notions, occurring in 39% of responses. Same behavior was coded when participants spoke generically about shared properties or behavior without specifying what was meant by these terms. For example, Participant 34 wrote: "I tell my students that two mathematical objects are "the same" if they have the exact same properties. That is, if you know how one of them behaves, then you know how the other behaves." While this explanation highlights what should be the same, attributes and behavior of two objects, it is not clear what those attributes or behavior are. Similarly, indistinguishable, mutual replaceability, and equivalence were used almost as synonyms for sameness, largely when addressing mathematical sameness in general. For example, Participant 72 wrote: "It depends on the situation! In a given situation, it means that for all purposes related to that situation, the two entities are indistinguishable." This response seems to suggest mathematical sameness is about being similar enough to put objects in the same class for a given situation.

Other participants highlighted preservation of operation or structure when describing mathematical sameness. References to operation-preservation and structure-preservation largely seemed to refer to the homomorphism property (or its analogue in another branch of math). For example, Participant 157 described sameness in algebra in a way that seems to refer to isomorphism, with operation-preservation describing the homomorphism property: "For groups and rings, there must be a one-to-one onto map (i.e., a bijection) that preserves the operation(s)." Participant 16 used structure-preservation in a similar way when describing mathematical sameness: "We identify structures we think are important or that somehow characterize the object, and declare that things are the same if they share those structures. This is usually formalized by the concept of a structure-preserving map." This example seems to imply that "structure" is the homomorphism property or an analogous idea in another discipline. Note, even though "structure" is often used to refer to objects like groups, rings, and other algebraic objects, this did not seem to be the structure intended in "structure-preservation."

Relabeling and matching were used to describe sameness in a way that aligned with isomorphism in algebra contexts. For example, Participant 196 described sameness in abstract algebra as "Effectively you can relabel one using the labels from the other and the structures will be maintained (i.e., we can define an isomorphism between them)." Essentially, we can interpret this "sameness" as being able to interchange the names of elements without fundamentally changing the underlying object. Although similar to relabeling, we described responses as "matching" when more emphasis was placed on the act of pairing elements in different objects instead of looking at structural relationships as a whole. For example, Participant 172 wrote:

> You know that they are the same if you can prove that an isomorphism between them exists. There are many ways to do this: for instance, for small enough groups, finding a way to organize the elements in two groups so that their group tables correspond suffices.

Note that the act of matching elements in Cayley tables was highlighted as a verification method, which has a qualitatively different sense from changing names to create a new, equivalent group.

## Discussion and Future Work

Mathematicians talk about sameness in a variety of ways. Notions of sameness are present throughout math, have many names, and have many subtle distinctions. Sameness connects to much of what we do in math, whether verifying objects all satisfy a shared definition or verifying objects are the "same" up to a sufficient level, which permits reasoning about the object in a new way. Mathematicians' responses highlighted this balance through the general views of sameness as context-bridging and context-dependent. Mathematicians emphasized that "sameness" needs to be clearly defined. However, this emphasis on qualifying sameness leads to questions of motivation: is emphasis placed on precise definitions because this is a mathematical norm, or do mathematicians view students as too inexperienced to determine what needs to be the same in a given context? Potentially, both factors are important. Certainly, evidence suggests students can attend to unintended aspects when considering sameness (Melhuish & Czocher, 2020).

While not as common as the philosophical codes, a number of ways of thinking of sameness applicable to proof settings were highlighted. Though students often consider specific properties that hold for isomorphic groups (Dubinsky et al., 1994; Leron et al., 1995), it is less clear how much they examine context-limited notions like mutual inverses in algebra and mutual subsets in analysis. This aligns with the need to consider content-based differences when examining students' understanding of proof (Dawkins & Karunakaran, 2016).

Mathematicians used many types of informal language to discuss sameness, including same behavior, structure-preservation, and relabeling. While these types of language have been used for isomorphism and homomorphism previously, seeing them employed in a broader participant pool is encouraging (Rupnow, 2021). Furthermore, the use of informal language sets up an interesting contrast with the emphasis on the context-dependence of sameness. On one hand, a majority of respondents highlighted the importance of context to making sense of sameness and emphasized context clarity in discussing sameness. However, informal language, by definition, introduces ambiguity into statements. Future research should examine how mathematicians balance precise language with their informal language, both for themselves and when teaching.

Furthermore, what students take away from informal expressions is unclear. Hausberger (2017) specifically notes that "structures" are often left undefined, leaving students to determine for themselves what precisely is preserved in a given context. Although mathematicians in this study seemed to refer to relationships between elements when referring to structure-preservation, their uses of the word structure were not always clear. Recall Participant 3's statement "Sameness usually means preservation of structure, and what abstract algebra is studying [is] structure". The "preservation of structure" seems to refer to relationships between elements, but "what algebra is studying is structure" could be interpreted in an object (e.g., groups) sense.

This study suggests mathematicians have a variety of philosophical approaches to sameness, including math as both context-dependent and context-bridging. Furthermore, mathematical sameness can refer to discipline specific notions like isomorphism or general notions related to proof, such as logical equivalence. Finally, mathematicians invoked a number of informal ideas to describe sameness, despite emphasizing the importance of precision in mathematics.


## Acknowledgments

This research was funded by a Northern Illinois University Research and Artistry Grant to Rachel Rupnow, grant number RA20-130.

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