# The Effectiveness of a Professional Development Video-Reflection Intervention: Mathematical Meanings for Teaching Angle and Angle Measure

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*Abstract: In response to calls for more investigations of teachers' mathematical meanings for teaching (MMT) and the need to improve US mathematics teachers' and students' mathematical meanings, I investigated mechanisms for advancing post-secondary instructors' mathematical meanings and teaching practices in the context of their teaching precalculus mathematics using a research-based curriculum. This report presents results from clinical interviews with graduate teaching instructors (GTIs) to illustrate the impact of a video-reflection intervention on teachers' MMT and image of effective teaching practices. More specifically, I present two GTIs' expressed meanings for angle measure and image of effective teaching before and after attending a professional development seminar where a trigonometry video-reflection intervention was used. I conclude by hypothesizing that video-reflection interventions may orient teachers toward reflecting on the degree to which they are impacting student thinking by providing examples of high-quality teaching (Musgrave & Carlson, 2017) interactions.*

*Keywords:* mathematical meanings for teaching, video-reflection intervention, quantitative reasoning

## Introduction and Review of Literature

Although many math educators have investigated teachers' *mathematical knowledge for teaching* (MKT) (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005), and how it relates to students' learning (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004; Hill et al., 2008), few researchers have investigated teachers' *mathematical meanings for teaching* (MMT). More concerning is the results of the few studies that have. As one example, Byerley and Thompson (2017) reported that most secondary mathematics teachers they studied provided formulaic or chunky descriptions of slope in contrast to thinking of slope as a rate of change that involves a multiplicative comparison between changes in two quantities. Other researchers who have investigated teachers' MMT have reported similar findings highlighting the impoverished nature of US teachers' mathematical meanings (Musgrave & Carlson, 2017; Yoon & Thompson, 2020; Thompson & Milner, 2018).

A teacher's meanings constitute her image of the mathematics she teaches (Thompson, 2013), her pedagogical decisions, and the language she uses to cultivate similar images in students' thinking (Thompson & Thompson, 1996), ultimately influencing the meanings her students are able to develop (Thompson, 2016). Prior research has shown that teachers' MMT develop in concert with their development of coherent mathematical meanings and reflection on the effectiveness of their instructional practices in supporting students' development of coherent meanings (Silverman &Thompson, 2008). Others have shown that teachers' meanings for foundational mathematical ideas can become more productive if they are engaged in interventions designed to support their development of coherent mathematical meanings (Musgrave & Carlson, 2017). However, little is known about the mechanisms for advancing instructors' teaching practices. Some researchers' have proposed implementing video-based professional development seminars (video clubs) (Sherin & Han, 2004; Seidel et al., 2009; Sherin & van Es, 2009; van Es, 2012; van Es et al., 2014). However, this research has been focused on the K-12 level and it has been shown that simply bringing teachers together does not ensure community development of desirable practices and norms (van Es, 2012). As such, investigating interventions for developing teachers' MMT and mechanisms that result in teachers' engagement in reflection on practice is of vital importance. This report addresses the following question: what is the impact of video-reflection intervention on teachers' MMT and image of teaching?

## Theoretical Perspective

The problematic nature of trigonometry teaching and learning has been documented by many (Moore et al., 2016; Tallman, 2015; Thompson, 2008; Thompson et al., 2007; Weber, 2005). Researchers (Hertel and Cullen, 2011; Moore, 2012, 2014; Tallman, 2015; Thompson, 2008) have responded by leveraging quantitative reasoning (Thompson, 1990, 2011), to support students in conceptualizing angle measure. Within the theory of quantitative reasoning, a *quantity* is a quality of an object or situation that one conceives of as admitting a measurement process. *Quantification* is the process by which one assigns numerical values to qualities (Thompson, 1990). A quantity's *value* is the numerical result of a quantification process. It is crucial to distinguish between *numerical operations* and *quantitative operations.* Numerical operations are used to calculate a quantity's value, whereas a quantitative operation is the conception of two quantities taken to produce a new quantity (Thompson, 1990). Although quantitative reasoning has been shown to support students in conceptualizing angle measure, students and teachers often develop meanings that do not involve this way of thinking (Thompson, 2008).

Tallman and Frank (2018) described a productive meaning for angle measure grounded in quantitative reasoning. Namely, "a quantitative understanding of angle measure involves identifying an attribute of a geometric object to measure and conceptualizing a unit with which—and process by which—to measure it" (Tallman & Frank, 2018, p. 5). Many students and teachers recognize that measuring an angle involves quantifying the openness between two rays that meet at a common vertex. "However, the "openness" of an angle is a quantity only for those who can specify a particular attribute of a geometric object to measure, identify an appropriate unit of measure, and envision a measurement process" (e.g., iteration, multiplicative comparison) (Tallman and Frank, 2018, p.5). Thus, one may conceptualize measuring the openness of an angle by measuring the length of the arc of a circle subtended by the angle whose vertex lies at the center of the circle in a unit proportional to the circle's circumference that contains the subtended arc.

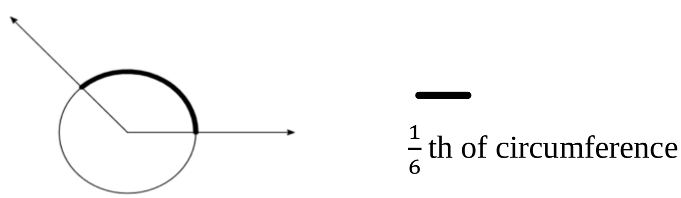


*Figure 1. Angle Measure*

For example, we could measure the angle in *Figure 1* by multiplicatively comparing the subtended arc length to the length of the circle's circumference. As such, the angle would have a measure of 1/3 in units of the circle's circumference. Namely, the subtended arc length is 1/3 times as large as the length of the circle's circumference. One could also imagine measuring the angle's openness in *Figure 1* by multiplicatively comparing the length of the subtended arc to the length of $1/6^{th}$ of the circle's circumference. The angle would have a measure of 2 in units of $1/6^{th}$ of the circle's circumference since the length of the subtended arc is two times as large as $1/6^{th}$ of the circle's circumference. Thus, a productive understanding of angle measure involves conceptualizing angle measure as a measurement process that defines a multiplicative relationship between the subtended arc length and some unit proportional to the circle's circumference (Moore, 2014).

## Methods, Subjects, and Data Collection

This study aimed to investigate the degree to which teachers' MMT and image of effective teaching shifted as a result of engaging in video-reflection tasks at a weekly professional development seminar. To accomplish this goal, I conducted two rounds of clinical interviews (Clement, 2000) with three graduate teaching instructors (GTIs) and recorded their engagement in a video-reflection task during a professional development seminar. At the time of the study, the GTIs were in their first semester of teaching using research-based *Pathways to Pre-Calculus* materials. The first round of clinical interviews took place a week before the instructors taught a lesson on angle and angle measure and attended a PD seminar where they discussed the video-reflection tasks. The second round of clinical interviews took place within three days of the instructors' lesson and the PD seminar. The analysis of the clinical interviews occurred in three phases as described in Simon (2019). The first phase involved my listening to the interviews to generate hypotheses about the teachers' ways of thinking. The second phase involved a more in-depth, line-by-line conceptual analysis aimed at describing aspects of the teachers' MMT for ideas of angle and angle measure. The third and final phase of this analysis involved identifying themes in the conceptual analysis to characterize the teachers' MMT.

## The Intervention

The GTIs involved in this study participated in a professional development seminar that met once per week for one semester. During the $11^{th}$ week of the semester, the GTIs were engaged in a video-reflection intervention designed to support instructors in implementing the *Pathways to Pre-calculus* curriculum with fidelity. The angle measure video-reflection intervention contains six video clips of an instructor teaching a lesson on angle and angle measure. The selected clips include segments of teaching in which the instructor demonstrated strong MMT, decentering abilities (Steffe & Thompson, 2008), teaching practices that involve speaking with meaning (Clark et al., 2008), mathematical care (Hackenburg, 2010), attending to student thinking, and conceptually-oriented explanations (Thompson & Thompson, 1996). The video-intervention also includes a set of questions that correspond to each of the video clips. We designed these questions to promote teachers' reflection on their mathematical meanings for angle and angle measure and practices they might engage in to support students' construction of coherent meanings for the ideas central to the lesson.

As one example, the trigonometry video-reflection intervention includes a clip of an interactive class discussion surrounding a task that asks students to determine which of three given angles has the largest measure. In this clip, the instructor poses multiple questions to students to elicit their meaning for angle measure and what they consider to be an appropriate unit for measuring angles. The video reflection intervention includes the following reflection questions for instructors to discuss after watching this clip.

1. In the video clip, the teacher asked, "what are you guys looking at when you're talking about angle measure?" Discuss with your partner (1) what you think the teachers' purpose for this question may have been? (2) What might be a more productive way the

teacher could have posed the question “what are you guys looking at when you’re talking about angle measure”?

2. At the end of clip two, the instructor draws three circles such that the angle’s vertex lies on the center of each circle. What might have been the teachers’ purpose for doing this? Discuss with a partner, what you might do next in your own class to help students develop the conception that an appropriate unit of angle measure is proportional to the circumference of a circle.

We hypothesize that the video-reflection interventions may orient teachers toward reflecting on the degree to which they are impacting student thinking by providing imagery of the ways of thinking the teachers need to develop in their students, and by providing examples of teachers listening to students’ thinking and asking questions to gain insight into students’ thinking.

## Results

This section includes results from the pre/post interviews that highlight the GTI’s expressed meanings for angle measure and image of effective teaching. The participants were asked to complete multiple tasks designed to elicit their meanings for angle and angle measure in each of the clinical interviews. This section shares two instructors, Emani and Maliya’s, responses to the angle measure task in *Figure 2.*

In class, a student tells you that the measure of a given angle is 1.2 inches. Is this an appropriate measure for an angle? Why or why not?

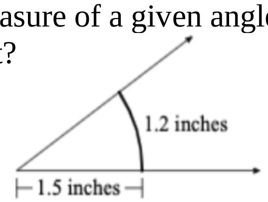


*Figure 2. Angle Measure Task- Interview 1*

## GTIs’ Pre-Intervention Meanings for Angle and Angle Measure

### Angle Measure as parts of a whole- the story of Emani

Emani’s expressed meaning for angle measure before engaging in the video intervention is best exemplified by her response (see Excerpt 1) to the task in *Figure 2.* It is noteworthy that Emani was the only instructor who consistently expressed that angle measure is a measure of openness between the two rays formed by the angle. Emani also expressed that she could determine the measure of an angle by counting how many units of measure are cut off by the angle’s rays. Thus, it appears as though Emani conceived of angle measure as fitting parts (some unit) into a whole (subtended arc).

Excerpt 1: *Emani’s expressed meaning of angle measure prior to intervention.*

*Emani:* Angle measure should be measuring how much openness we can measure between the rays. And measuring a length is not the same as measuring an amount of openness. So, the ways that jump out at me right now for measuring this angle, I could make up other ones, I guess, but are degrees or radians. Measuring in degrees would be like finding out how many 1/360ths [highlights portion of subtended arc] are cut off by the angle marked with 1.2 inches [traces over subtended arc].

While Emani was able to identify the attribute (length of the subtended arc) and a process by which to measure it (fitting pieces into a whole), she struggled to describe an appropriate unit for measuring the angle and why it was appropriate. In Excerpt 2, we can see that Emani correctly identified degrees and radians as appropriate units of measure, and she was also able to create her own unit of measure *diametans.* However, Emani was unable to provide a meaningful explanation for why degrees, radians, and *diametans* were appropriate units of measure.

Excerpt 2: *Emani's discussion of an appropriate unit for measuring angles*.

*Interviewer*: So earlier, I am going to take you back to the beginning when you first read this. You immediately read this task and you said, "oh like it could be degrees, or it could be radians, or I could make something up." So, what did you mean by "I could make something up"?

*Emani*: Hmm, umm, so, radians are measured by looking at the radius of the circle. Degrees are measured by looking at some portion of the circumference of the circle centered where the rays meet. So, I think we can measure any angle, or come up with any kind of angle measure as long as we're measuring in terms of some portion of the… like anything constant with the circle? Like we could come up with an angle measure instead of radians that is like *diametians.* So, you measure with the diameter of the circle. So, radians look at how many radius lengths are being subtended, are being cut off [indicates with hands that she is laying lengths next to each other] by the angle. So, we can do the same thing but with the diameter.

**Angle Measure- the result of a quantitative operation: the story of Maliya**

Maliya consistently expressed a meaning for angle measure as a comparison of the "distance you walk along that portion of the circumference that you marked with the two rays to the entire circumference of the circle" (see Excerpt 3).

Excerpt 3: *Maliya's expressed meaning for angle measure prior to intervention.*

*Maliya:* Ok this is harder than I thought it would be. An inch is not a unit that I would use to measure an angle because an angle would be umm, so an angle is going to measured in some unit relative to the circumference of the circle. So, to be more specific, if I have a circle and I want to know the angle between two rays, then I would say that you can measure an angle by comparing how far I have walked from the horizontal ray to this slanty ray (traces over the terminal ray) compared to the size of the circumference of the circle.

Thus, it appears as though Maliya conceived of angle measure as a multiplicative comparison of the subtended arc length and the circumference of the circle for which the angle is centered. I interpret Maliya's meaning for angle measure to be that of a quantitative operation as she consistently identified (see Excerpt 4) the result of the multiplicative comparison as the measure of a specific attribute (length of the subtended arc) in a particular unit (the circumference of the circle).

Excerpt 4: *Maliya's discussion of the angle measure task.*

*Maliya:* So, the endpoint of the angle (points to the angle's vertex) is going to be the center of the circle. So, I can say that 1.5 inches is the radius of the circle. So, if I want to know the circumference of the circle that's going to be 2 pi times as large as the length of the radius, so 1.5 times 2 pi. So, the 1.2 inches, I would say that is a measure of the length of that portion of the circle's circumference that is taken up by the arc between those two rays. But the measure of the angle would be a comparison of the length of the arc, 1.2 inches, divided by 1.5 times 2 pi inches.

*Interviewer:* So, what does 1.2 divided by 1.5(2pi) represent?

*Maliya:* To me it represents the measure of the length of the arc between the two rays given in the diagram in units of the circumference of the circle with radius 1.5 inches.

While Maliya expressed a quantitative meaning for angle measure, her descriptions of angle measure were limited to using the circumference as a unit of measure. Maliya did not recognize that she could measure the length of the subtended arc in radius lengths or any other unit proportional to the circumference of the circle.

**GTIs' Post-Intervention Meanings for Angle and Angle Measure**

Overall, the instructors' expressed meaning for angle measure remained consistent with their expressed meaning before engaging in the video-reflection intervention and teaching a lesson on angle and angle measure. Maliya continued to describe angle measure as a multiplicative comparison of the subtended arc and the length of the circumference. Emani's meaning for angle measure also remained consistent with the meaning she expressed before engaging in the video-reflection intervention. However, Emani was more closely able to provide a quantitative description of angle measure that involved identifying the attribute (length of the subtended arc) of the angle, and a process by which to measure it (fitting pieces into a whole), and a unit by which to measure the length of the subtended arc (fractional portions of the circumference). Although Emani was able to identify a unit of measure, she struggled to express what a correct unit of measure is meaningfully and was aware of her difficulty doing so. As one example, Emani expressed in the second interview, "I need to work on my own understandings and a better way to say this."

**GTIs' Image of Effective Teaching Before the Intervention**

The instructors were asked to describe their image of good/effective teaching before and after engaging in the video-reflection intervention. The instructors' image of good teaching expressed in the first interview is shown in *Table 1.* It is noteworthy that both Emani and Maliya expressed that good teaching involved attending to student thinking, awareness of how students are feeling in relation to their learning, and ensuring students are comfortable contributing to class discussions.

*Table 1. Teachers' response to prompt: describe your image of good/effective teaching.*

| Teacher | Sample Excerpts from Clinical Interview 1 |
| --- | --- |
| Emani | Good teaching is trying to understand where…how the students are understanding what is going on. I try to ask questions of my students like as we work through problems to try to find out how they're thinking so I can try to address how they're thinking about the problem. |
| Maliya | I would say, asking students questions really trying to get students input. It's a good practice that I tried to implement that exemplifies good teaching and I think really trying to help students to think for themselves and articulate their thinking as much as possible. |

**GTIs' Image of Effective Teaching After the Intervention**

During the post interview, Maliya and Emani described asking questions to elicit student thinking as an essential quality of a good teacher. However, Maliya's response is particularly interesting as she described multiple practices that were exemplified in the video clips and reflective discussions included in the video-reflection intervention (see Excerpt 5). One goal of the video-reflection intervention is to promote a reflective discussion about the instructors' goals for students' learning. After participating in the video-reflection intervention, Maliya described "having clear expectations for what you want the students to learn" as an essential quality of good teaching. Maliya also expressed the importance of having a discussion-oriented classroom and an iterative conversation with students. Maliya's expression of the importance of having an iterative conversation with students is also interesting as the video-reflection intervention provided examples of a teacher eliciting, attending to, and responding productively to student thinking. Moreover, Maliya also mentioned, "really trying to speak with meaning in the class." It is noteworthy that *speaking with meaning* emerged as a practice that all three instructors identified as important throughout their second clinical interview.

Excerpt 5: *Maliya's response to the prompt "describe your image of good/effective teaching"*

*Maliya:* I do think having clear expectations for what you want the students to learn what you want the students to be able to master in terms of content and having clear expectations for assignments and assessments is important. But I do really like having the discussion-oriented classroom…and I'm really digging into their thinking when class is discussion oriented. I also do like having the visualization, especially because this course is so focused on quantitative reasoning that can be one of the most powerful ways to try to convey meaning to students and help students kind of convey their meaning…and just kind of like having an iterative conversation of, like, revising, okay, so this is what I think you said and then the student can correct you if that's not an accurate representation of what they said. So sometimes doing that can be useful to you like reiterating what you think the students said that can also help with students feeling comfortable revealing their thinking and really making clear and really trying to speak with meaning in the class.

## Conclusions and Discussion

Overall, both of the instructors' meanings for angle measure remained relatively consistent with their meanings before engaging in the video-reflection intervention. Following their teaching of the section and their participation in a PD seminar, the instructors were more likely to distinguish between and angle and the measure of the angle. However, Maliya and Emani did not advance in their ability to describe a more general way of thinking for measuring an angle using a unit (such as a circle's radius length) that is proportional to a circle's circumference.

While the instructors' meanings for angle and angle measure did not shift after engaging in a video-reflection intervention, their image of effective teaching did. The instructors' participation in the video-reflection intervention appeared to have an impact on their image of teaching, including their recognition of the importance of *speaking with meaning* and posing questions to reveal student thinking. It is noteworthy that the one-hour video-reflection intervention had little impact on the instructors' conceptions of angle and angle measure. However, this result is not surprising as it supports others' findings (Thompson and Thompson, 1996; Thompson, 2016; Musgrave & Carlson, 2017; Tallman and Frank, 2018) that shifting teachers' conceptions is a gradual process that requires sustained and repeated opportunities to reason in new ways. Although we have developed video-reflection interventions for many units in the *Pathways to Pre-calculus* curriculum, this was the first time they were used with GTIs. In future semesters we plan to engage the GTIs more video-reflection interventions to investigate how teachers' MMT and image of effective teaching shift over the course of a full semester. As we continue to develop and refine the video-reflection interventions, we plan to include more video clips that show teachers' promoting students' engagement in quantitative reasoning since prior research has shown that teachers' attention to/focus on referencing quantities, including the starting point and directionality of the measurement, is useful for advancing student learning (Moore & Carlson, 2012; Joshua, 2019).

## References


Ball, D. L. (1990). The Mathematical Understandings That Prospective Teachers Bring to Teacher Education. *The Elementary School Journal*, *90*(4), 449–466. JSTOR.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. Paper presented at the 2002 Annual Meeting of the Canadian Mathematics Education Study Group, Edmonton, AB.

Ball, D. L., Hill, H. C., & Bass, H. (2005). *Knowing Mathematics for Teaching*. 10.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior*, *48*, 168–193. https://doi.org/10.1016/j.jmathb.2017.09.003

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. Journal for Research in Mathematics Education, 33(5), 352. https://doi.org/10.2307/4149958

Castillo-Garsow, C., (2010). Teaching the Verhulst Model: A Teaching Experiment in Covariational Reasoning and Exponential Growth.

Clark, P. G., Moore, K. C., & Carlson, M. P. (2008). Documenting the emergence of “speaking with meaning” as a sociomathematical norm in professional learning community discourse. The Journal of Mathematical Behavior, 27(4), 297–310. https://doi.org/10.1016/j.jmathb.2009.01.001

Clement, J. (2000) Analysis of clinical interviews: Foundations and model viability. In Lesh, R. and Kelly, A., Handbook of research methodologies for science and mathematics education (pp. 341-385). Hillsdale, NJ: Lawrence Erlbaum.

Confrey, J., & Smith, E. (1995). Splitting, Covariation, and Their Role in the Development of Exponential Functions. *Journal for Research in Mathematics Education*, *26*(1), 66–86. https://doi.org/10.2307/749228

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. Cognition and Instruction, 25(4), 439–478. doi:10.1080/07370000701632397

Hackenberg, A. J. (2010). Mathematical Caring Relations in Action. Journal for Research in Mathematics Education, 41(3), 236–273.

Hertel, J., & Cullen, C. (2011). Teaching trigonometry: A directed length approach. In L. R. Wiest & T. Lamberg (Eds.), Proceedings of the 33rd annual meeting of the North American chapter of the inter- national group for the psychology of mathematics education. Reno, NV: University of Nevada, Reno.

Hill, H. C., & Ball, D. L. (2004). Learning Mathematics for Teaching: Results from California's Mathematics Professional Development Institutes. *Journal for Research in Mathematics Education*, *35*(5), 330–351. JSTOR. https://doi.org/10.2307/30034819

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction*, *26*(4), 430–511. https://doi.org/10.1080/07370000802177235

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement. *American Educational Research Journal*, *42*(2), 371–406. https://doi.org/10.3102/00028312042002371

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing Measures of Teachers' Mathematics Knowledge for Teaching. *The Elementary School Journal*, *105*(1), 11–30. https://doi.org/10.1086/428763

Joshua, Surani A. (2019). Conceptualizing and reasoning with frames of reference--three studies. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. The Journal of Mathematical Behavior, 31(1), 48–59. https://doi.org/10.1016/j.jmathb.2011.09.001

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. Mayes & L.L. Hatfield (Eds.), Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context (pp. 75–92). Laramie, WY: University of Wyoming.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. Journal for Research in Mathematics Education, 45(1), 102–138.

Moore, K. C., LaForest, K. R., & Kim, H. J. (2016). Putting the unit in pre-service secondary teachers' unit circle. Educational Studies in Mathematics, 92(2), 221–241.

Musgrave, S., Hatfield, N., & Thompson, P. (2015). Calculus students' meanings for difference. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, pp. 335-348. Pittsburgh, PA: RUME.

Musgrave, S., & Carlson, M. P. (2017). Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching. *The Journal of Mathematical Behavior*, *45*, 137–149. https://doi.org/10.1016/j.jmathb.2016.12.011

Seidel, T., Stürmer, K., Blomberg, G., Kobarg, M., & Schwindt, K. (2011). Teacher learning from analysis of videotaped classroom situations: Does it make a difference whether teachers observe their own teaching or that of others? Teaching and Teacher Education, 27(2), 259–267. https://doi.org/10.1016/j.tate.2010.08.009

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. Teaching and Teacher Education, 20(2), 163–183. https://doi.org/10.1016/j.tate.2003.08.001

Sherin, M. G., & van Es, E. A. (2009). Effects of Video Club Participation on Teachers' Professional Vision. *Journal of Teacher Education*, *60*(1), 20–37. https://doi.org/10.1177/0022487108328155

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*(6), 499–511. https://doi.org/10.1007/s10857-008-9089-5

Simon, M. A. (2019). Analyzing Qualitative Data in Mathematics Education. In K. R. Leatham (Ed.), *Designing, Conducting, and Publishing Quality Research in Mathematics Education* (pp. 111–122). Springer International Publishing. https://doi.org/10.1007/978-3-030-23505-5_8

Simon, M., & Tzur, R. (2004). Explicating the Role of Mathematical Tasks in Conceptual Learning: An Elaboration of the Hypothetical Learning Trajectory. *Mathematical Thinking and Learning*, *6*, 91–104. https://doi.org/10.1207/s15327833mtl0602_2

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and Essential Elements. In Research design in mathematics and science education (In R. Lesh & A. E. Kelly (Eds.), pp. 267–307). Hilldale, NJ: Erlbaum.
Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*. Free Press.
Tallman, M. (2015). *An Examination of the Effect of a Secondary Teacher's Image of Instructional Constraints on His Enacted Subject Matter Knowledge*. https://doi.org/10.13140/rg.2.1.4871.5363
Tallman, M. A., & Frank, K. M. (2018). Angle measure, quantitative reasoning, and instructional coherence: An examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-018-9409-3
Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University.
Thompson, P. W., & Thompson, A. G. (1994). Talking about Rates Conceptually, Part I: A Teacher's Struggle. *Journal for Research in Mathematics Education*, *25*(3), 279. https://doi.org/10.2307/749339
Thompson, P., & Thompson, A. (1996). Talking About Rates Conceptually, Part II: Mathematical Knowledge for Teaching. *Journal for Research in Mathematics Education*, *27*(1), 2–24.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education, 1, 45–64.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), New perspectives and directions for collaborative research in mathematics education WISDOMe Monographs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming Press.
Thompson, P. (2013). In the Absence of Meaning…. In *Vital Directions for Mathematics Education Research* (pp. 57–93). https://doi.org/10.1007/978-1-4614-6977-3_4
Thompson, P. W. (2016). Researching mathematical meanings for teaching. In English, L., & Kirshner, D. (Eds.), Handbook of International Research in Mathematics Education (pp. 435-461). London: Taylor and Francis.
Thompson, P. W., & Milner, F. (2018). Teachers' meanings for function and function notation in South Korea and the United States. In H.-G. Weigand, W. McCallum, M. Menghini, M. Neubrand & G. Schubring (Eds.), The Legacy of Felix Klein (pp. 55-66). Berlin: Springer.
van Es, E. A. (2012). Examining the development of a teacher learning community: The case of a video club. *Teaching and Teacher Education*, *28*(2), 182–192. https://doi.org/10.1016/j.tate.2011.09.005
van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A Framework for the Facilitation of Teachers' Analysis of Video. *Journal of Teacher Education*, *65*(4), 340–356. https://doi.org/10.1177/0022487114534266
Weber, K. (2005). Students' understanding of trigonometric functions. Mathematics Education Research Journal, 17(3), 91–112.
Yoon, H., & Thompson, P. W. (2020). Secondary teachers' meanings for function notation in the United States and South Korea. Journal of Mathematical Behavior, online