Sameness Connections across Mathematics

Brooke Randazzo
Northern Illinois University

Rachel Rupnow
Northern Illinois University

Helping students see conceptual connections between content areas is important for the development of flexible understanding, yet research on ways in which notions of sameness throughout math can be connected is limited. This study examines survey responses from mathematicians on the relevance of sameness to topics in abstract algebra and connections between sameness in algebra and other courses. Common connections are highlighted as are themes in the types of connections provided.

Keywords: Isomorphism, Homomorphism, Abstract Algebra, Instructional Practice, Sameness

A common motivation behind initiatives in math education is to see how content can be made relevant to students, especially pre-service teachers. Recent initiatives in undergraduate mathematics have largely approached this task by connecting K-12 content to advanced courses like analysis (e.g., Wasserman et al., 2018) and abstract algebra (e.g., Suominen, 2018) or by examining defined mathematical topics such as functions (e.g., Melhuish & Fagan, 2018; Melhuish et al., 2020) and binary operations (e.g., Melhuish et al., 2020) at various levels. In this paper, we take a slightly different approach by considering how an undefined concept, sameness, appears across math. Specifically, we address the following research question: What connections do algebraists see between sameness in abstract algebra courses and other math content?

Literature Review and Theoretical Perspectives

From the beginning of students' mathematical experiences, they are introduced to the notion of mathematical equality. Students are directed to attend to equal counts of objects already in Kindergarten (National Governors Association Center for Best Practices, 2010, p. 11) and continue utilizing notions of equality throughout their mathematical careers. Despite the centrality of equality to mathematics, students do not always have a conceptual understanding of equality. Multiple studies have examined students' conceptions of equality and have connected students' modes of understanding to their ability to do algebra (e.g., Alibali et al., 2007; Kieran, 1981). Specifically, students' ability to manipulate expressions while maintaining equality is aided by understanding equations as two expressions that relate to each other in a specific way instead of viewing the equal sign as an indication to compute something (Alibali et al., 2007). As students mature mathematically, notions of sameness are expanded from equality to congruence in geometry and verifying equivalence in trigonometric expressions (e.g., $\sin^2 \theta + \cos^2 \theta = 1$).

As students reach college mathematics, more complex ideas of equivalence arise such as isomorphism and homomorphism in abstract algebra. While some research has examined students' methods of verifying isomorphism (e.g., Dubinsky et al., 1994; Leron et al., 1995; Melhuish, 2018), recent work has examined students' understanding of isomorphism and homomorphism in other ways. For example, Hausberger (2017) considered how students understand "structure-preservation" of homomorphisms and other authors have examined isomorphism and homomorphism through the lens of conceptual metaphors (e.g., Melhuish et al., 2020; Rupnow, 2017; 2019).

However, work on mathematicians' views of isomorphism and homomorphism remains limited. Weber and Alcock (2004) highlighted algebraists' notions of isomorphism as meaning groups were “essentially the same” (p. 218) or that groups being isomorphic meant “one group was simply a re-labelling of the other group” (p. 218). When speaking with algebra instructors, we have found four major categories of metaphors used to describe isomorphism or homomorphism: mappings, sameness, sameness/mapping, and formal definition (Rupnow, 2021). Nevertheless, the prevalence and extent to which algebraists connect isomorphism or homomorphism to other notions of sameness in math and how such connections are made in instruction are unknown.

A pair of theoretical lenses for examining how knowledge in different contexts might be connected comes from Lobato (2012). In her paper, Lobato contrasts the mainstream *cognitive perspective on transfer* with *actor-oriented transfer* (AOT) and highlights places in which each can be effective. The mainstream perspective characterizes transfer as “how knowledge acquired from one task or situation can be applied to a different one” (Nokes, 2009, p. 2) and generally takes an observer’s perspective: an outside party determines whether or not transfer of some “desired” knowledge occurred. In contrast, AOT characterizes transfer as “the generalization of learning, which can be understood as the influence of a learner’s prior activities on her activity in novel situations” (Lobato, 2012, p. 233). Thus, AOT instead takes an actor’s perspective, where any expansion in the use of knowledge is viewed as transfer, even if the “desired” knowledge was not what transferred. Both of these perspectives can be useful in examining which concepts of sameness mathematicians transfer from lower-level classes through abstract algebra.

Methods

Data was collected via a survey that was sent to every 4-year college or university math department that offers abstract algebra in the United States. This survey addressed how algebraists think about sameness in general and in specific mathematical contexts. Participants were 197 mathematicians from 173 institutions who had taught at least one abstract algebra or category theory course in the last five years. For this paper, only the following three open-ended questions regarding sameness connections will be discussed:

1. Which abstract algebra topics lend themselves to deepening students’ understanding of mathematical sameness?
2. What connections do you see between sameness in abstract algebra and in prior math courses?
3. What sameness connections between abstract algebra and other courses do you (or could you) help students make when teaching?

These were the last non-demographic questions asked in the survey. Prior questions had asked about participants’ broad views of sameness in math as well as their specific views about isomorphism and homomorphism. We noted that many participants answered questions 2 and 3 similarly or referenced their answer to question 2 in their answer to question 3. For this reason, these two questions were grouped together for analysis.

Two independent coders used thematic analysis (Braun & Clarke, 2006) which included multiple iterations of coding (Anfara et al., 2002). First, responses were open-coded using descriptive coding (Saldaña, 2016) for words indicating any connection to sameness. For questions 2 and 3, an effort was made to also code the particular course in which the sameness connection was made. Responses could receive multiple codes. Any discrepancies between the two coders were discussed before coming to an agreement. Second, codes were organized according to theme. Third, responses coded within a given theme were examined as a group for consistency within the theme and to determine trends in general ideas captured by particular codes. Finally, the above steps were repeated to refine the codes and assure consistency.

Results

Connections to Sameness in Abstract Algebra

This section involves codes related to the following survey question: "Which abstract algebra topics lend themselves to deepening students' understanding of mathematical sameness?" The list of codes and number of participants who were coded in such a way are presented in Table 1.

Table 1: Connections to Sameness in Abstract Algebra

Category	Code	Participants
Equivalence Relations	Equivalence relation	29
	Bijection	7
	Isomorphism	113
	Automorphism	3
	Homomorphism	47
	Quotients	37
	Modular Arithmetic	13
	Group Actions	5
	Fundamental Theorems	23
Organization/Schema	Category Theory	9
	Classification Theorems	20
Broad Statements	No connections	5
	Everything	13

Equivalence relations. The majority of participants answered this question by referencing an equivalence relation or sets of equivalence classes. Several mentioned *equivalence relations* explicitly, e.g., "Equivalence classes. They are the essence of elemental sameness. Once [students] understand that different elements can be the same (under the right lens), then it comes naturally to view structures as being the same." Codes related to examples of equivalence relations were broken into three subcategories: renaming, structure loss, and relating the two.

Renaming. This subcategory involves codes related to a simple renaming of a structure: *bijection* (renaming sets), *isomorphism*, and *automorphism* (renaming algebraic structures). Although there is some debate in the literature about the level of equivalence implied by a renaming (Mirin, 2019), renaming (and its cousin relabeling) have previously been documented as ways mathematicians describe isomorphism (Rupnow, 2021; Weber & Alcock, 2004).

The most common code to appear in answers to this question was *isomorphism*, being mentioned by 113 participants (57%). Some suggested that isomorphism and sameness were linguistically interchangeable in the context of abstract algebra, e.g. "I want my students to think of 'the same' as 'isomorphic.'" Indeed, many highlighted its centrality to the study of algebra:

> Arguably, the concept of isomorphism is central to everything in the course. The whole point of teaching it to undergraduates is to help them see that the numbers and operations they grew up with are actually instantiations of more universal properties of a wide range of mathematical objects...that some of their intuition about the way numbers work translates into theorems about other cool objects.

Structure loss. This subcategory involves codes related to sameness that may involve some structure loss: *homomorphism*, *quotients*, *modular arithmetic*, and *group actions*. For example, many participants mentioned *homomorphisms* to refer to sameness of parts of structures. Often, participants noted that the sameness here lies in the relationship to quotients: “Homomorphism is useful in that it says something about being structure-preserving and you can find sameness in the fibers.” In fact, one participant discussed the idea of *quotients* as an opportunity to explore and make precise various notions of sameness:

> In examining the fundamental set theoretic construction of passing from a set with an equivalence relation to the quotient set (set of equivalence classes) we have occasion to pay attention to different but related concepts: “two elements of X are the same”, “two elements of X are equivalent”, “two equivalence classes in X are equal.” This should help the students to understand that there are different concepts related to “sameness” that need to be understood precisely.

Relating the two. This subcategory includes the code *fundamental theorems*, which refers to what are commonly called “the isomorphism theorems” and/or “the fundamental homomorphism theorem.” This family of theorems provides a bridge between two types of sameness discussed above: isomorphism and homomorphism. The theorems involve using a homomorphism between groups to create an isomorphism between related groups and using isomorphisms to reinterpret quotients, thereby permitting both loss of some structure and a renaming of subparts to occur. For example, one participant mentioned “the idea of the first isomorphism theorem, that [homomorphisms] are the same concept as quotient groups.”

Organization/schema. Codes in this category give a way of classifying ideas about sameness in abstract algebra. For example, *category theory* provides general language with which to talk about structures and morphisms beyond just abstract algebra and can be used to define sameness more generally: “I introduce a little bit of category theory… to emphasize the ‘context independent’ way to say when two objects are the same.” The code *classification theorems* is also categorized here because these types of theorems give a way to organize algebraic structures according to isomorphism classes or other broader classes.

Broad statements. A small number of participants were coded as *no connections* when they indicated that the idea of sameness was not well-defined enough for them to identify it with concepts from abstract algebra, e.g., “There is no such thing as mathematical sameness outside of specific context.” In contrast, others were given the code *everything* when they indicated that all of the concepts in abstract algebra are related to sameness in some way, e.g., “All of them, in a sense. In all of algebra, the point is to distinguish between features that matter and features that don’t.” This idea of sameness being dependent on context was something that was frequently mentioned in answers to the questions featured in the next section.

Connections to Sameness in Other Courses

This section involves codes based on answers to the following survey questions: (1) “What connections do you see between sameness in abstract algebra and in prior math courses?” (2) “What sameness connections between abstract algebra and other courses do you (or could you) help students make when teaching?” The list of codes and number of participants who were coded in such a way are presented in Table 2.

Equivalence relation. The explicit mention of *equivalence relations* appeared more often in reference to other math courses than in answers to the abstract algebra specific question. This is likely because, as some participants pointed out, equivalence relations are often students’ first experience with sameness beyond literal equality:

> The best reference point I can think of that most students will have had is the idea of an equivalence relation - we sometimes use an equivalence relation to regard two objects as the “same” (or related) even if they are not precisely the same object in the domain of the

relation. But abstract algebra takes that view of objects and applies it to entire structures, which themselves are composed of many objects.

Codes for specific types of equivalence relations are split up into the two subcategories below, which are analogous to the renaming and structure loss subcategories from the previous section.

Table 2: Connections to Sameness in Other Courses

Category	Code	Participants
Equivalence Relations	Equivalence relation	49
	Bijection	26
	Identities	4
	Equality	14
	Arithmetic—representations of numbers	21
	Geometry—congruence/isometry	31
	High school algebra—algebraic manipulation	5
	High school algebra—log/exponential function	10
	Calculus—limit	7
	Calculus—change of variables	3
	Calculus—coordinates	2
	Linear algebra—change of basis	7
	Linear algebra—isomorphism	26
	Graph theory—isomorphism	9
	Differential geometry—diffeomorphism	6
	Topology—homeomorphism	20
	Functions--one-to-one/onto	6
	Modular arithmetic	23
	Geometry—similarity	17
	High school algebra—graph transformations	3
	Calculus—derivatives and antiderivatives	18
	Differential equations—solving differential equations	4
	Linear algebra—transformations/homomorphisms	25
	Linear algebra—similar matrices	7
	Linear algebra—RREF	4
	Topology—continuous maps	3
	Analysis—almost everywhere	3
Language/ Organization/Schema	Category theory	13
	Proof techniques	7
Broad Statements	No connections	10
	Deepen understanding of sameness	53

Renaming/change of perspective. This subcategory involves codes related to a renaming of an object or a change of perspective that does not change the object itself. Some codes included here are direct analogs to abstract algebra's isomorphism that appear in the undergraduate curriculum, such as *linear algebra—isomorphism*, *graph theory—isomorphism*, or *topology—homeomorphism*. These are all fairly obvious connections to make, though some pointed out that the notions of isomorphism that students tend to see prior to abstract algebra are much more straightforward. For example, one participant noted: "Sometimes there is a graph theory unit in a discrete mathematics course, but somehow that's still too intuitive. (Isomorphic graphs don't really look like different objects...someone just drew them a little differently.)"

Others reached further back in the curriculum and included examples from high school (e.g., *geometry—congruence/isometry*) or elementary school (e.g., *equality, identities*). An interesting connection that many made was to *arithmetic—representations of numbers,* where participants pointed out that equivalent fractions can be reinterpreted as sets of equivalence classes. Still others made connections to renamings that provide a clear change of perspective or additional insight, such as *linear algebra—change of basis*: "For example, most students are familiar with change of basis before taking abstract algebra, and understand that using different coordinates (in calculus also) makes the problem look different, but is mathematically equivalent."

Structure loss/change of object. This subcategory involves codes related to sameness that may involve some structure loss. For example, many made the connection to *linear algebra—transformations/homomorphisms*, which are a direct analog of homomorphisms in abstract algebra. Others mentioned *modular arithmetic* as one of the first places where students see more abstract notions of sameness and suggested it as a starting point for equivalence classes and quotients: "Modular arithmetic is really 'the same' as arithmetic on quotients (groups, rings, etc), so whenever we have a quotient, we should be able to define a modular arithmetic." One interesting connection was to *calculus—derivatives and antiderivatives,* where participants noted an antiderivative is actually an equivalence class of functions that all have the same derivative.

Language/organization/schema. Codes in this category give a way of organizing and thinking about ideas of sameness within and across disciplines, such as *category theory*. One participant noted that "most (possibly not all) [connections to sameness] are captured by morphism-related notions in a suitable category." The code *proof techniques* is also categorized here because the language and techniques used to write mathematical proofs provide ways to think about mathematical objects across disciplines.

Broad statements. A large number of participants (53 participants, 27%) indicated that abstract algebra is an opportunity for deepening understanding of sameness, although many did not explicitly state that they make attempts to deepen this understanding with their students. Some participants coded here said that the ideas of sameness in abstract algebra generalized or made more precise earlier concepts of sameness, e.g., "I think that students have to take a somewhat more liberal view of 'sameness' in abstract algebra than in most prior math courses." Others stated that the idea of sameness is a thread that shows up throughout math, only with changes to what count as relevant characteristics:

> Emphasizing that "sameness" is context-dependent: what is and what is not "the same" depends on what we consider to be important for our purposes. I encourage [students] to keep an eye out for this in other courses and contexts, and to try to figure out what [are] the "essential characteristics" that are important in any given subject.

Again, some participants indicated that they do not focus on sameness or are not sure what connections they can make to sameness. Others indicated that they believe that abstract algebra is students' first encounter with the idea of sameness, saying things like "I don't think we have previously covered sameness in a way that would shed light on the topic."

Discussion

Participants tended to equate sameness with concepts based on equivalence relations. Isomorphism was the most frequently mentioned abstract algebra concept that was linked to sameness, and homomorphism was a distant second. While this may be partially explained by prior questions in the survey directly asking about isomorphism and homomorphism, the fact that so many participants voluntarily cited them again and cited related concepts in other subjects suggests some importance placed on isomorphism and, to a lesser extent, homomorphism.

Codes classified as renaming, including *geometry—congruence/isometry*, *linear algebra—isomorphism*, *bijection*, and *arithmetic—representations of numbers*, were the most common connections to other courses. While it is not surprising that individuals would highlight analogues of isomorphism, especially in another form of algebra, the variety of specific types of renamings and the number of connections to K-12 topics were surprising.

Codes in the structure loss subcategory were mentioned less often than those in the renaming subcategory. This is likely because codes there refer to similar objects that may have some properties or partial structure in common, but do not share as many properties or structure as the renaming examples. These included connections to transformations in linear algebra, modular arithmetic (number theory/discrete math) or antiderivatives (calculus).

A few mathematicians resisted making sameness connections, suggesting that isomorphism was a "richer" type of sameness than students experienced previously. However, this opinion was expressed by a clear minority of respondents. In contrast, roughly a quarter of respondents viewed abstract algebra as deepening students' understanding of sameness and made connections to a variety of prior math content, suggesting a view that sameness permeates mathematics.

While some participants only highlighted "obvious" examples of sameness, many made connections across multiple branches and/or levels of mathematics, suggesting sameness is relevant to much of the math curriculum. This highlights the importance of students transferring prior knowledge and experiences with sameness to the new contexts of isomorphism and homomorphism in abstract algebra (e.g., Lobato, 2012). However, the extent to which students transfer this knowledge independently or based on explicit classroom experiences remains an open question. Prior research in analysis (e.g., Wasserman et al., 2018) suggests that pre-service teachers are more likely to connect fully formed explanations to future teaching than general concepts. However, they also acknowledge that their pre-service teachers may not have "developed the web of connections to appreciate the conceptual insights that the explanations in our study (and perhaps real analysis in general) may provide" (p. 87). We suggest that taking a larger grain size, such as examining sameness across mathematics, may provide a way to create the "web of connections" necessary to support students' transfer of big ideas across courses.

Conclusions and Future Work

In this study, we used actor-oriented transfer (AOT) to examine what mathematicians transferred across contexts with respect to sameness (Lobato, 2012). In accordance with mainstream transfer (Lobato, 2012), we also laid a foundation for future research by soliciting mathematicians' views of what "should" transfer across contexts. Specifically, future research can examine what notions of sameness students are already transferring independently or based on instruction. Furthermore, some mathematicians did not claim to explicitly connect notions of sameness in abstract algebra while others did. Future research could examine how connections are made in class and see if some methods are more effective for encouraging transfer than others. In so doing, we hope to help math majors make meaningful connections across their studies and better prepare students to apply their knowledge flexibly in their future careers.

Acknowledgments

This research was funded by a Northern Illinois University Research and Artistry Grant to Rachel Rupnow, grant number RA20-130.

References

Alibali, M. W., Knuth, E. J. Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221–247. doi: 10.1080/10986060701360902

Anfara, V. A., Brown, K. M., & Mangione, T. L. (2002). Qualitative analysis on stage: Making the research process more public. *Educational Researcher, 31*(2), 28-38. doi:10.3102/0013189X031007028

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*(3), 267-305.

Hausberger, T. (2017). The (homo)morphism concept: Didactic transposition, meta-discourse, and thematisation. *International Journal of Research in Undergraduate Mathematics Education*, *3*(3), 417–443. doi: 10.1007/s40753-017-0052-7

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Lobato, J. (2012). The actor-oriented transfer perspective and its contributions to educational research and practice. *Educational Psychologist, 47*(3), 232–247. doi: 10.1080/00461520.2012.693353

Melhuish, K., Ellis, B., & Hicks, M. D. (2020). Group theory students' perceptions of binary operation. *Educational Studies in Mathematics, 103*(1), 63–81.

Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*(1), 9–38.

Melhuish, K. & Fagan, J. (2018). Connecting the Group Theory Concept Assessment to core concepts at the secondary level. In N. H. Wasserman (Ed.), *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 19–45). Cham, Switzerland: Springer Nature. https://doi.org/10.1007/978-3-319-99214-3_2

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60*.

Mirin, A. (2019). The relational meaning of the equals sign: A philosophical perspective. In Weinberg, A., Moore-Russo, D., Soto, H., & Wawro, M. (Eds.). (2019). *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 783-791). Oklahoma City, Oklahoma.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* National Governors Association Center for Best Practices, Council of Chief State School Officers: Washington D.C.

Nokes, T. J. (2009). Mechanisms of knowledge transfer. *Thinking and Reasoning, 15*(1), 1–36.

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education*, (pp. 259–273), San Diego, CA.

Rupnow, R. (2019). Instructors' and students' images of isomorphism and homomorphism. In Weinberg, A., Moore-Russo, D., Soto, H., and Wawro, M. (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education.* (pp. 518-525), Oklahoma City, OK.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62*(2), 100867. doi: 10.1016/j.jmathb.2021.100867

Saldaña, J. (2016). *The coding manual for qualitative researchers.* Thousand Oaks, CA: SAGE Publications, Inc.

Suominen, A. L. (2018). Abstract algebra and secondary school mathematics connections as discussed by mathematicians and mathematics educators. In N. H. Wasserman (Ed.), *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers,* Research in Mathematics Education (pp. 149–173). Cham, Switzerland: Springer Nature. doi: 10.1007/978-3-319-99214-3_8

Wasserman, N., Weber, K., Villanueva, M., & Mejia-Ramos, J. P. (2018). Mathematics teachers' views about the limited utility of real analysis: A transport model hypothesis. *Journal of Mathematical Behavior, 50*, 74–89. doi: 10.1016/j.jmathb.2018.01.004

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2-3), 209–234.