Conceptualizing and Representing Distance on Graphs in Calculus: The Case of Todd

Erika David Parr, Middle Tennessee State University
Rob Ely, University of Idaho
Cynthia Piez, University of Idaho

Numerous calculus concepts rely on the ability to conceive of and represent distances in the Cartesian plane. Yet, research has found that undergraduate students, even those who have studied calculus, may not readily do so (Parr, 2020). This study reports on findings from the development of a hypothetical learning trajectory aimed at supporting students in representing distance in the Cartesian plane. We hypothesized that this skill would require (a) the ability to represent a distance on a number line as a difference and (b) drawing on the Cartesian connection to represent a horizontal or vertical distance in terms of x and in terms of y. We analyze the types of reasoning used by a Calculus I student while working on these tasks, as well as obstacles she faced relative to these two target understandings.

Keywords: Calculus, Graphical Representations, Distance, Hypothetical Learning Trajectory

Consider the following task. A generic point (x, y) is indicated on a graph of $y = \sqrt{x-1}$. A horizontal segment is drawn from this point to the line $x = 2$. Represent the length of the segment (a) in terms of x and (b) in terms of y.

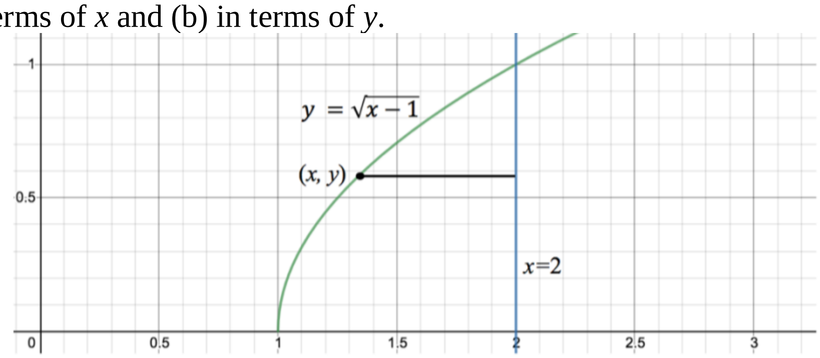


Figure 1. Distance between two curves task.

The skill reflected in this task is the ability to represent distances in the Cartesian plane in terms of input and output values of a graphed function. This skill is crucial in Calculus; it is used when creating integrals that represent areas bounded by curves and volumes of rotation, and also for representing a difference quotient for a function in the definition of the derivative. This skill relies on conceptualizing a point's coordinates (x, y) as describing the horizontal and vertical (signed) distances between the point and the origin. Yet, research has shown that students may not conceive of coordinates of points in the plane as expressing distances when reasoning about related expressions (Parr, 2020). In fact, when the second author gave this task to the students in his Calculus I class, only 17.4% of students answered both (a) and (b) correctly, and only 37% answered one of the parts correctly. This was, alarmingly, <u>after</u> the chapters about finding areas between curves and volumes of solids of revolution using integration.

Our broader goal for this study is to investigate how students can learn this important skill. We conducted a teaching experiment (Steffe & Thompson, 2000) with a hypothetical learning trajectory (Simon & Tzur, 2004) involving a sequence of tasks intended to develop this skill. This study focuses on several elements of student reasoning that emerged during that teaching experiment.

Theoretical Background

In this study, we focus on two target understandings that we hypothesized to be important for correctly solving tasks like the one above. The first is conceptualizing a difference of the form $b - a$ as representing the distance (along a one-dimensional continuum) between a and b. Since this understanding is grounded in interpreting notation, it helps to frame it, and the other related concepts we found in the study, using the language of semiotics (Barthes, 1957). A sign is made of a symbol (signifier) and a signified. The signified can be a mental object or another mark or collection of marks (which itself could be a signifier of something else). An interpretation is then the association between the symbol and the signified. With this in mind, we describe this first understanding as a *composed magnitude interpretation* of distances on number lines.

In the context of representing a segment's length, we define a composed magnitude interpretation as treating an expression of the form "*b–a*" as representing the following object: the length (a measurable attribute) of the segment measured from a to b. This requires the construction of the mental object of a quantity (e.g., Thompson, 1994). This interpretation of *b–a* is consistent with Parr's (2020) description of a *magnitude* interpretation. We extend this definition to include the notion that the distance is *composed* of the difference of two distinct distances. The *b* is the distance from the origin to one endpoint, the *a* is the distance from the origin to the other, and *b–a* is the difference between these distances, thus the length of the segment. Finally, we contend that this interpretation involves appealing to a difference model of subtraction (5–2 is the difference between 2 and 5) as opposed to the more prevalent take-away or counting down model of subtraction (5 take away 2 leaves you with 3) (Saxe et al., 2013).

The second understanding we hypothesized to be important for tasks like the one above involves using the Cartesian connection (Knuth, 2000) to represent the distance between a general point on a function's graph and the *x*- and *y*-axes, in terms of *x* and in terms of *y*, flexibly. The Cartesian connection refers to the relationship: points lie on a graph if and only if the coordinate pairs (*x*, *y*) satisfy its equation. To use this relationship to represent distances in terms of *x* and *y* involves conceiving of a coordinate point (*x*, *y*) as a multiplicative object (Saldanha & Thompson, 1998) of a pair of distances from the axes, rather than as the name of a location. Although interpreting coordinates as a pair of distances from the point to respective axes may seem elementary, undergraduate students may instead interpret coordinates nominally, as labels for locations on an axis or the graph (David et al., 2019; Parr, 2020).

In terms of these two understandings, our research question for this particular study is: What types of reasoning emerge, and what obstacles are encountered, when students engage in tasks meant to develop their ability to express distances on graphs in the Cartesian plane?

Methods

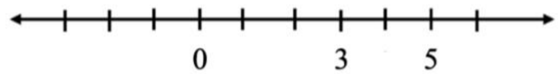
As part of a hypothetical learning trajectory (HLT) (Simon & Tzur, 2004), we designed six tasks and corresponding learning goals to support students in coming to conceive of and express distances on graphs such as the one in Figure 1. In fact, this exact task was given as both a pre-assessment and as the final task in the sequence. We aimed to sequence the tasks such that each subsequent task built off the previous, with a single new conception or step added. The goal of the first task of the sequence is to develop the composed magnitude interpretation of a difference. As such, this task asked students to represent and label a difference (5–3) on a number line by first drawing a segment from the origin of length 5 (a), a segment from the origin of length 3 (b) (Figure 2).

1. Draw the segments described using the x-axes provided.

a. Draw a segment with a length of 5 on the x-axis below, beginning at 0.

b. Draw a segment with a length of 3 on the x-axis below, beginning at 0.

c. Draw a segment to represent the difference 5 – 3 on the x-axis below and label it "5 – 3."



d. What is the length of the segment you drew in part c.?

Figure 2. A portion of Task 1 of HLT to develop composed magnitude interpretation.

The tasks progressed to asking students to represent various difference expressions as segments, involving whole numbers, integers, rational numbers, and variables as distances on both horizontal (Task 1) and vertical number lines (Task 2). Task 3 reversed this and provided students with horizontal or vertical segments, asking students to express the length of the given segment as a difference. Task 4 advanced from number lines to the Cartesian plane, giving students a generic point (x, y) and asking them to construct a segment from each axis to the point and label its distance. This task sought to make the Cartesian connection explicit in students' reasoning. Task 5 was similar, giving a point on a graph of a linear equation, but this time asking students to label distances both in terms of the input variable x and the output variable y (See Figure 3). Task 6 asked the same questions as Task 5 for a non-linear equation.

We conducted a teaching experiment (Steffe & Thompson, 2000) using our HLT with four groups of two students and one individual student (whose classmate was absent) from an undergraduate Calculus I course in Fall 2020 in their recitation section (held remotely) during the first week of the course. Students worked through the activity in breakout rooms on Zoom, with the researchers observing and offering assistance only if students could not resolve an issue in a timely manner. When these obstacles arose, we asked a series of questions intended to scaffold the necessary understandings we intended for the task. Often, these scaffolds took the form of asking a similar question with a numerical value rather than a variable, clarifying the intention of the task, or referring students back to previous tasks. We then conducted thematic analysis of the transcripts of these videos to identify and characterize students' interpretations relative to the development of conceptions we intended.

Results

Our goal in this study was to identify types of reasoning students use and obstacles they face in coming to conceptualize and represent distance between two curves. Using the case of one student, Todd, we describe two related obstacles that arose in many of the groups of students, which required additional support from the interviewers beyond the tasks to overcome. The first is related to students' conceptions of distance of a segment of endpoints "*a*" and "*b*." The second is related to students' ability to express the same distance in different ways using relationships between input and output variables via the Cartesian connection.

Interpretations of Distance of a Segment

While working with students through these tasks, we noticed three distinct types of student interpretations of distance emerge that were distinct from the composed magnitude interpretation we were targeting:

1. *Endpoints interpretation.* Students using an 'endpoints' interpretation of distance referred to a segment's length, when asked, as "from 0 to *a*" or "from *a* to *b*," giving the endpoints of the segment. In these instances, students reported where the segment began and ended but seem to be describing the segment itself rather than imagining a quantity of distance.

2. *Template interpretation.* Students using a template interpretation of distance may represent the segment length as 0–*a*, but "–" symbol is considered a dash, not the operation of subtraction. In this way, they apply a template, transferring "from *a* to *b*" into "*a*–*b*" or "*b*–*a*" (and may not be sure when or why the order matters). If they use "*b*–*a*," it serves as a (single) sign for the following object: the segment with endpoints at b and a.

3. *Magnitude interpretation.* Students using a magnitude interpretation refer to the distance of the segment as *b*–*a*, which signifies a quantity, consistent with Parr (2020). It is a sign for the following object: the (measurable attribute) of the length of the segment. Students give the difference "*a*–0," and treat the segment's distance as a measurable attribute of the segment with a numerical value that corresponds to the difference. This interpretation requires *b*–*a* to represent a single amount, even if it is unsimplified or unsimplifiable. This interpretation does not necessarily draw upon the subtraction as representing the difference between two magnitudes of length *a* and *b* themselves; in this way it differs from the composed magnitude interpretation we were targeting (described earlier in the theoretical background section).

We note that the simplification of the difference is available, as a small syntactic move, to students who use the magnitude interpretation. The distance "*a*–0" is the same as "*a*" and the distance "5–2" is the same as "3." On the other hand, students who did not use the magnitude interpretation did not readily recognize that "*a*"is the distance from 0 to *a*, nor readily invoke that it is the same as *a*–0.

We use the case of Todd (female student who chose her own pseudonym) to illustrate how the distinction in these interpretations became apparent to us. As context, we describe an earlier exchange between Todd and her classmate Walter that occurred with Task 2. Todd had asked whether the expressions were "additive" or "all just differences." Walter "assumed" they were all "differences," but was unclear about whether to interpret the "–" symbol as subtraction or a dash. She said,

> So I thought it was like above [Task 1] where we, I guess that's where I might have gone wrong up above…I assumed…that it was asking from 3.5 *to* 2, like the subtract was, or the dash was *to* instead of, or is it supposed to be a subtraction thing?

In this instance, Walter appears to be using an endpoints interpretation of the distance between 2 and 3.5 as "from 3.5 to 2" and is unsure about how to interpret the "–" symbol. This endpoints interpretation becomes relevant in Todd's later work on these tasks. While working on Task 5a, Todd was at first confused about the intention of the question, and thought *x* indicated a location at *x*=0 on the y-axis, the left endpoint of the horizontal segment. To clarify, Interviewer 1 explained that the point represented a generic point (*x*, *y*) on the line. This exchange follows:

Int 1: Is there a general way of saying how long that [green horizontal segment from the *y*-axis to a generic point on the line *y*=2*x*+1] is in terms of just the, the point itself which is (*x*, *y*)?

Todd: I feel like there is but I, I don't really have any ideas on that like I just maybe again it would just kind of <u>go back to the whole zero to the *x*-coordinate</u> like because <u>if I don't</u>

want to say exactly what length it is, I just go from like let's say every time, doesn't matter what length the segment...it just it stayed on the *y*-axis and it never went past it you know so it always in relation to the *x*-axis be at zero if I didn't want to say the length of the horizontal segment, I would simply just say from zero to *x* wherever that *x* and the (*x*, *y*) would be.

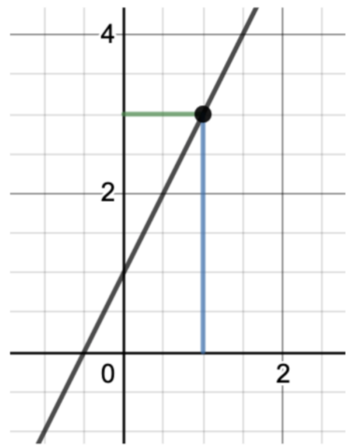
5. The graph of $y = 2x + 1$ is shown to the right.

a. Represent the length of the horizontal segment (green) in terms of *x*. (Your expression should have "*x*" in it).

b. Represent the length of the horizontal segment (green) in terms of *y*. (Your expression should have "*y*" in it).

c. Evaluate your expressions at the point where *x* equals 1, and confirm that you get the same result from a. and b.

d. Represent the length of the vertical segment (blue) in terms of *y*.



e. Represent the length of the vertical segment (blue) in terms of *x*.

Figure 3. Task 5 of the HLT- represent the horizontal and vertical segments in terms of x and y

In this exchange, Todd first appeals to an endpoints interpretation, stating the endpoints of the segment rather than the length of that segment. We also infer that Todd recognized that the length of the segment was varying ("doesn't matter what length the segment is"), and that the location of *x* was varying ("wherever that *x*…would be"). However, Todd did not use a magnitude interpretation, and was even unable to express the distance as the difference *x*–0, in spite of completing the previous 4 tasks, with the help of prompting. When asked the same question with a numerical value, Todd was quick and confident to answer correctly:

Int 1: How about this: If you have a segment that goes from zero to two, how long is it?
Todd: It's two
Int 1: If you have a segment that goes from zero to five how long is it?
Todd: 5
Int 1: If you have a segment (Todd: Sorry) go ahead
Todd: I was just gonna say so for this one from zero to it's clearly one, it would be one.
Int 1: Yeah, and so if it goes from zero to *x* how long is that?
Todd: *x*
Int 1: That's it.
Todd: So it's just *x*? For [part] a?

At first glance, it might seem that the only obstacle in Todd's reasoning is in moving from a specific value (2 or 5) to an abstract value (*x*), or that Todd had difficulty conceiving of *x* as a varying value. However, on the graph, Todd is treating *x* as a varying *location*, an endpoint of a segment of varying length. It was step of associating "*x*" with the varying *length* of the segment that appeared to be the obstacle. Thus, we take this obstacle as more than an issue of moving between a numerical value and varying value in the symbolic register—this required a shift in how she was interpreting how variables were assigned in the graphical register. Because of this obstacle, Todd reverts to an endpoints interpretation rather than a magnitude interpretation.

Signifying a Variable Quantity in Multiple Ways

Another obstacle we noticed in two groups of students was in expressing a variable distance in another way (e.g., in terms of *y*, rather than *x*). This obstacle became apparent when students did not interpret a given equation of a function as relating distances within a graph; in other words, these students failed to make the "Cartesian connection" (Knuth, 2000) between the given equation and the graph. As an example, we continue to look at Todd's work on Task 5, part b. Todd explains her thoughts for this question, which asks her to represent the same segment's length, now in terms of *y*.

> *Todd*: Okay so this is where my head is going but …when I look at it in relation to numbers uh you know the blue line would be y right so that one's three *x* is one…I think if it is kind of like <u>*y* over 3</u> would represent the horizontal segment because it's *y* divided by three so three divided by three would be one and since *x* is one *y* over three…
>
> *Int 2*: Oh, it sounds like you're connecting in some way the one and the three or the *x* and the *y*, right? (Todd: yeah) Is there a way that *x* and *y* are connected that were told…?
>
> *Todd*: Well are you talking about like the, the linear function or are you I don't know um
>
> *Int 2*: So yeah, what does the linear function tell you about *x* and *y*?
>
> *Todd*: (*reading equation*) $y=2x+1$, and then when I think about *x* and *y* in relation to the graph it tells you how it's going to look, so the slope would be 2, the *y*-intercept would be 1, yeah.

In this exchange, Todd recognizes the need to represent the horizontal distance from the *y*-axis to the graph in terms of *y*, and she attempts it with $y/3$. Todd uses the static distances of the point at (1,3) to devise this relation and appears to go through the process of how to operate on 3, a stand-in for *y*, to obtain 1, the stand-in for *x*. Todd knows what the task is asking of her, but does not use the given equation $y=2x+1$. When directed to the equation, she does not seem to think this gives her the relation she was searching for earlier with her conjecture of $y/3$. Instead, the equation represents instructions for the visual features of the graph ("how it's going to look") including values of the slope and *y*-intercept.

As the discussion continues, Todd does see the equation as useful (perhaps because she's been directed to) when asked to find lengths at specific non-variable values.

> *Int 2*: Would there be a way to for you to figure out what the horizontal length would be at that point where y would be 19?
>
> *Todd*: Yes
>
> *Int 2*: How would you do that?
>
> *Todd*: uh you would do the, basically find, I don't want to call it the inverse, but you would plug 19 in for *y* in the original equation and do $2x$ plus 1, subtract 1, to be 18 equals $2x$, so then your *x* would be 9, right? 18 over 2, 9? Yeah so, your horizontal segment would be 9 if your *y* segment was 19.
>
> *Int 2*: So if I gave you any *y*-value would you be able to figure out the horizontal length then?
>
> *Todd*: Yes, oh, okay, okay wait, wait, wait, wait uh actually I don't know if I'm going the right way but I'm gonna do it anyway two yeah okay so if I were to plug in 3 for *y* there *x*

would wind up being 1 which is what it is, I guess I just <u>I'm getting kind of confused on how to include *y*</u> in there what I just kind of would I uh *y* equals two *x* plus one could I do *y* minus one equals two *x* divided by two, so *y* minus one over two equals *x*, would that be the horizontal component trying to express like…?

Int 2: okay so would that work um that *y* minus one over two?

Todd: Oh my god, it would work, it would work, okay.

It is notable that Todd quickly and confidently answers the question of what the horizontal length would be when *y* is 19. When given a specific *y*-value that is not shown, she sees the equation as useful for finding the corresponding *x*-value, and she recognizes it also to be a length. But she had not immediately generalized this process to work for *y* as a variable length. She retraces the same operations she used on 19, this time using *y*, to get the equation's inverse relationship $x = \frac{y-1}{2}$, and is then confident in her response. We theorize that this general expression of the Cartesian connection was previously unavailable to her because it relies on interpreting *x* and *y* as variable magnitudes, an interpretation she has not previously developed.

Discussion

We reported two obstacles in coming to develop an ability to represent distance in the Cartesian coordinate system through examples from one student, Todd. Todd's endpoints interpretation of distance was an obstacle on Task 5 when using *x* to represent a varying length of the horizontal segment. Another obstacle for Todd was using the Cartesian connection to represent the same segment in terms of *y*. We conjecture that both of these obstacles are rooted in a lack of a composed magnitude interpretation of differences, yet more research is needed to investigate the relationship among these conceptions.

Further research is needed to better understand the development of the conception and representation of distance in the Cartesian plane to complete tasks like the ones in this study. For instance, future research may investigate how students come to develop a composed magnitude interpretation of differences. We intend to refine our tasks and continue to conduct teaching experiments to better characterize and support the cognitive processes involved in coming to understand a composed magnitude interpretation of differences.

We conclude by returning to the composed magnitude interpretation we described earlier as the foundational understanding involved in representing distances in the Cartesian plane. At the heart of the composed magnitude interpretation is the underlying notion that each value represented on a number line can be conceived of as a distance from 0. When we use a difference to represent a distance between two values, we are utilizing two distinct conceptions of a value *x*: (1) *x* is a location on a number line and (2) *x* gives the distance from the location on the number line called *x* and the origin. In practice, we as instructors may switch between these two conceptions seamlessly and without acknowledgment that they are fundamentally distinct conceptions. For instance, when we say that *b–a* gives the distance between *a* and *b*, we are treating *a* and *b* as locations, yet the reason that the difference gives that distance relies on unpacking *a* and *b* as also giving the distance from the origin to each of those locations. When we use *b–a* to represent this distance, we may take for granted the underlying mental actions involved with understanding why a difference yields this distance. The results of this study indicate that the process of developing a composed magnitude interpretation is non-trivial and should be considered central to the goals of teaching calculus conceptually.

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