A Preservice Teacher's Experience of Mathematical Research

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As part of a summer research project, an undergraduate named Anabella (pseudonym) and I (author) collaborated in efforts to prove an unsolved conjecture from the field of graph theory. Anabella is a Hispanic mother who one day desires to be an elementary or middle school teacher. The researcher collected and analyzed extensive data in order to understand what Anabella learned about the nature of mathematics through the research experience as well as how she perceived this new knowledge is relevant to her future as a teacher of mathematics. Anabella learned that her own mathematical ideas are valuable and reflected that it will be important to entertain and consider (rather than dismiss) the ideas of her future mathematics students. She also gained a new appreciation of the importance of mathematical communication.

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Scholars in mathematics education have found that elementary and middle school teachers often have naïve views of the nature of mathematics (Pair, 2017; Thompson, 1992; Watson, 2019). It is important for teachers in particular to have informed views of the nature of mathematics so that they may support their own students in developing informed rather than naïve conceptions of the nature of mathematics. The university, with its role in teacher development, plays a substantial role in providing opportunities for future teachers to enhance their perspectives of the nature of mathematics.

This study explores the opportunities afforded for a preservice elementary/middle school teacher to learn about the nature of mathematics through research mathematics—in this case, collaborating to prove an unsolved conjecture from the field of graph theory. Anabella (pseudonym) was a student in a content course for preservice elementary and middle school teachers that the author was teaching at a large public university in Southern California. Anabella stood out in the class for her willingness to share her solutions and engage with mathematics. On the first day of class she was drawing complete graphs to solve the handshake problem. Anabella also expressed interest in the discipline of mathematics after learning that there were still unsolved mathematics problems. The author wrote an application to fund Anabella with a student research assistantship and obtained approval from the university's institutional review board to collect data during the project in order to understand what Anabella learned about the nature of mathematics and its relevance to her future teaching through the research.

Literature Review

While many mathematics education scholars have argued that misconception of the nature of mathematics is a major problem in mathematics education (Boaler, 2016; Hersh, 1997; White-Fredette 2010), little research has been conducted that explicitly addresses how to nurture positive conceptions of the nature of mathematics. As Hudson and colleagues (2020) recently noted, "There is still much to learn about what students and teachers think about the nature of mathematics and how these views impact their work" (p. 55). There are some recent research articles and dissertations that address the conceptions of mathematics held by both preservice teachers and undergraduate mathematics majors. Kean (2012) developed a survey instrument designed to form a profile of teacher's conceptions of the nature of mathematics. However, she concluded that her first attempt needed much improvement; and that developing a good survey instrument would be more difficult than anticipated. To date, Kean's survey has not been used in subsequent research.

Pair (2017) argued that in order to conduct meaningful research into students' views of the nature of mathematics, the field would need to develop a list of goals for students' understandings of the nature of mathematics. Through an extensive data-driven project that included 1) interviews with mathematicians, educators, and mathematics majors; 2) collaboration with a pure mathematician in conducting mathematical research; and 3) experience co-teaching an inquiry-based transition-to-proof course with a mathematics educator, Pair developed the IDEA framework for the nature of mathematics. The framework drew inspiration from humanistic philosophy of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976) and emphasized the human discipline of mathematics and its dynamic, rather than its static nature. This framework will be discussed further in the theoretical framework section.

In studying the nature of mathematics conceptions of preservice elementary teachers, Watson (2019) found that many possessed a view of mathematics as "a static-unified body of knowledge" (p. 148). In interviews with some of these preservice teachers, she found that those who experienced mathematics learning through memorization and practice tended to view mathematics as a bag of tricks rather than a dynamic problem-solving discipline. However, there were some cases in which the preservice teachers from Watson's study experienced mathematics creatively. This led them to see mathematics as more of a creative, problem-solving discipline.

> Tabby's experience with one mathematics task and a teacher who valued her ideas in the classroom allowed Tabby to experience mathematics as a problem-driven dynamic discipline. Maggie, Tatum, Margot, and Millie also recounted experiences with teachers who allowed them to view mathematics as more than a bag of tools and instead as a discipline where creation and freedom to create could be valued. (p. 171)

Watson's findings align with those of Thompson (1992), who three decades earlier noted that that when a teacher views mathematics as "a discipline characterized by accurate results and infallible procedures" this "can lead to instruction that places undue emphasis on the manipulation of symbols whose meanings are rarely addressed" (p. 127). Watson's (2019) study suggests that providing students with opportunities for creativity through problem solving can lead to a more productive view and positive attitude towards mathematics.

While there is some older literature relevant that addresses the nature of mathematics (Dossey 1992; Lampert, 1990; Thompson, 1992), the topic still has yet to be systematically studied by mathematics education researchers (Hudson et al. 2020). While there are researchers who have recently begun to refocus on the nature of mathematics in their research, research findings are tentative. How do we best support students and future teachers in developing informed and healthy conceptions of the nature of mathematics? The object of the current study is to explore how an undergraduate research experience may impact a future teachers' understanding of the nature of mathematics.

Theoretical Framework

The foundational framework that guided this research was developed is the IDEA Framework for the Nature of Pure Mathematics (Pair 2017). The IDEA acronym refers to four characteristics of the nature of mathematics that are important for students to understand so that they are supported in developing a humanistic conception of the nature of mathematics. These characteristics are now described with a brief description. First, mathematical ideas are part of both our human cultural and individual personal **I**dentity; we inherit mathematical ideas from past mathematicians and cultures, and form our own personal mathematical ideas and techniques when engaged in doing mathematics. Second, mathematical knowledge is **D**ynamic and forever in the making; the history of mathematics is a story of creation and revision of ideas. Furthermore, new mathematical results are published every day. Thirdly, pure mathematics is an emotional and creative **E**xploration of ideas; students should have the opportunity to explore such ideas for their own sake in ways similar to that of mathematicians and understand that such exploration is essential to mathematics. Fourthly, mathematical **A**rgumentation is required for the social validation of mathematical knowledge; mathematicians validate and knowledge claims through verification and proof, leading to robust knowledge. The characteristics in this list may serve as learning outcomes for instructors wishing to promote a humanistic view of mathematics and as a conceptual framework to conduct research on students' views of the nature of mathematics. As these characteristics were inspired by my own experience conducting research mathematics, these were characteristics of mathematics that I anticipated Anabella would learn about in research.

Research Design and Methodology

Anabella is a Hispanic mother and was a traditional undergraduate senior at the time of the study. Anabella was compensated for participating in a summer assistantship; and one of her responsibilities was to keep a research journal in which she wrote about her experience and the new things she was learning about the nature of mathematics and how this new knowledge was relevant to her future as a teacher of mathematics. This journal was part of the data collected for the study. Additionally, every meeting was audio-recorded (with Anabella's permission as part of the study) in order to capture Anabella's experience so I could identify meaningful moments that contributed to her evolving conceptions of mathematics. The research experience lasted 2 months; and I collected approximately 60 hours of audio recordings of our meetings, and approximately 70 pages of Anabella's mathematical work and reflections in her journals. Other sources of data include photographs of our mathematical work and my mathematical notebooks in which I worked on the conjecture and took notes of Anabella's growth and expanding knowledge of the nature of mathematics. The goal in collecting so much data was to be able to later summarize the evolution of Anabella's conception of the nature of mathematics and describe in detail, through stories, the experiences that had the most impact on Anabella's growing understanding of the nature of mathematics and uncover how Anabella perceived the research experience was relevant to her future as a teacher.

After transcribing the data, I use qualitative approaches such as open coding and sorting in order to find the recurring themes in her reflections (Patton, 2015). The goal was to be able to paint a picture of the experiences Anabella had that led to transformations in her understanding of the nature of mathematics and also describe how Anabella believed those experiences and new understandings are relevant to her as a future teacher. All sources of data are brought together to tell stories about Anabella's experience and how that experience led to a change in her understanding of the nature of mathematics and its teaching and learning.

Results

I here describe the early stages of the research experience in detail so the reader can get an idea of the nature and scope of the mathematical research involved. After providing this background, I then provide a summary of the remainder of the experience and some of Anabella's takeaways from the research.

To begin the research, I shared with Anabella a paper from a mathematics journal which discussed our open problem—what is the chromatic number of graphs with girth 5 and no odd holes of length greater than 5? Anabella had several questions about the terminology and symbology of the paper, and on the first official day of our research collaboration, we discussed mathematical concepts such as girth, chromatic number, clique number, complete graph, cycle, chord, and hole. I asked Anabella to conjecture what was the relationship between the clique number and the chromatic number; and she conjectured that they were the same. We examined several graphs for which it was true that the chromatic number equaled the clique number, but ultimately found some examples for which this was not true. I took this as an opportunity to explain that it would be common for us to make and revise conjectures:

> So this is, I think, the first lesson we have had since we started. You are going to make conjectures that you are going to think are true; and you are going to find out they are not true, pretty quickly. It happens all the time [in mathematics] actually. I think that is important.

As we continued to think about the relationship between the clique number and the chromatic number, we discussed the journal article, which laid out the special conditions for which the chromatic number would be one more than the clique number. As we did this, Anabella reflected on how there are always new mathematical questions one can pursue. "This can go on forever and ever and ever," she said. Immediately, she got a new idea, a process of constructing new graphs from complete graphs which she described as "working backwards." Anabella took a complete graph, removed the "outside cycle," added vertices equal in number to the outside cycle, then connected the remaining vertices. See Figure 1 for some images of the process conducted on a few graphs. The Peterson graph is created through this process; and I believe Anabella invented the process based on her observations of the structure of the Peterson graph.

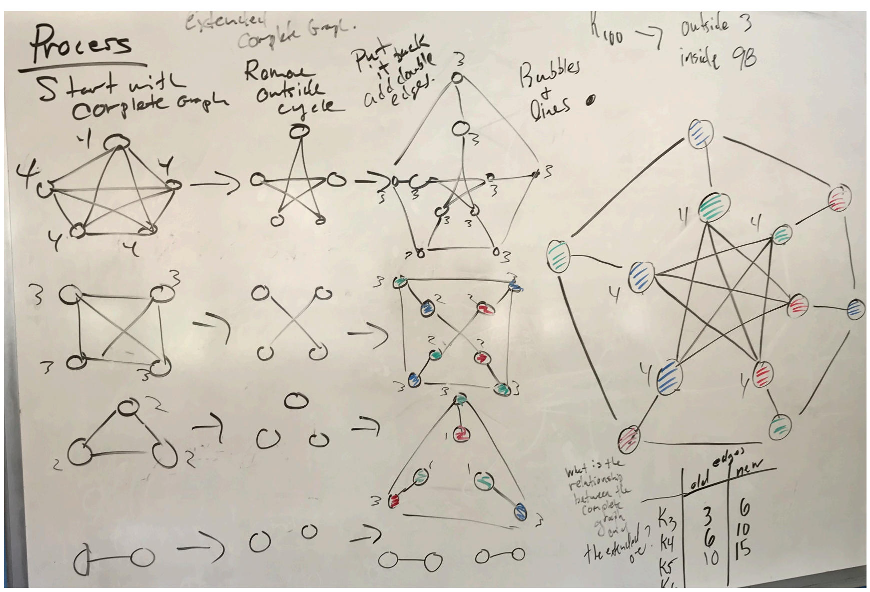

Figure 1. Anabella's Invented Process of Creating New Graphs from Complete Graphs

For the rest of the meeting, we worked with Anabella's process, creating new graphs from complete graphs, finding the chromatic number, and thinking about how many vertices and edges the original and new graphs would possess. I had never read about this kind of process in any graph theory text, and I was not sure if other mathematicians had done it before; I shared this with Anabella. She was excited at the prospect of exploring mathematics in a way that others had not. At the end of our first session she wrote,

> I am not fully convinced that no one else in the entire world, in its history, has ever came up with what we did today. However, if this is the case that only a few other people have ever done so, I feel a sense of great accomplishment because now I can say that I am one of those few. I have come up with this all on my own and it is only my first official day of work. I can't wait for what I will achieve tomorrow. Even if my biggest achievement is only in my head (or in this case, my notes). […] I have also begun to see the world (in the area of mathematics) as infinite. Just today, both Dr. Pair and I had tons of questions on each new thought/discovery we made. This is pretty exciting because I really enjoy being able to think "outside the box" and ask questions that would otherwise never even be considered taking a second look at. The sense of knowing things in general is fun to me and from today, I know it is also infinite. This is unlike any other subject because, let's say, with English or history, there are only so many things to know/learn.

On this first day, I believe that Anabella experienced the creative side of mathematics and learned more about its dynamic and infinite nature. She recognized that mathematical activity has no end, as there are always more mathematical questions one can ask and pursue. She enjoyed "thinking outside the box" which I have found students tend to associate with their first experiences of mathematical research (Pair & Calva, 2020).

On the second day of the research, Anabella, based on her own internet research, had many questions about concepts such as planarity and girth. The conjecture we were working on concerns the chromatic number of graphs with girth 5 and no odd holes of length greater than 5; Anabella created the name "high five class" for the set of such graphs. The properties of graphs in this class are generalized from the Peterson graph, and Anabella and I spent a substantial amount of time discussing the cycles of the Peterson graph in detail. We then proved an important (but simple) lemma for the high five class, that no graph in the class has a 7 cycle. We then discussed partitioning a graph in terms of levels, or distance from some arbitrary vertex. We took many graphs that we had already drawn and partitioned them into levels, also observing how the levelling process allowed us to systematically color the vertices of the graphs. We went through proof that each level is bipartite, and we also went through a proof that for any graph in the class with two levels, the chromatic number is 3. We accomplished a lot on Day 2, and after the day was over, Anabella wrote:

> My head hurts. I just crammed this new info into my brain in just a few hours. But I think we made more accomplishments today. For example, I helped Dr. Pair figure out that subgraphs in level 3 cannot connect. […] I learned about levelling and I was pretty fast at figuring those out. I learned that level 1 is always independent for any high-5 graph. The most important thing I think isn't understanding the new vocab, but understanding the concepts. I can always rename them like Dr. Pair lets me do. It is only my 2nd day with

Dr. Pair, but already have come to understand his dilemma, why he is stuck and cannot solve the problem. Still, we got many steps closer to solving it.

In this reflection, Anabella realizes what is important are the mathematical concepts, the ideas, rather than the vocabulary and the name of the concept. As she says, we can rename these. She often referred to vertices as “bubbles.” This is an important notion for teachers to understand—the name we give to mathematical concepts is not as important as understanding the concepts themselves. It is also enables one to stamp their own identity into mathematics as one chooses how to name mathematical objects.

On the beginning of the third day, Anabella took a lead in suggesting what we should work on in approaching the conjecture. She said, “I was thinking what if we focused on like the options, like focus on level 2. Because we know level 1 always has to be a color, and level 3 can alternate between two colors.” We spent most of the rest of the day imagining we had some even cycles in level 3, then looking at how the parents of those cycles were connected in level 2. Near the end of the session, I asked Anabella if she saw any similarities between the type of mathematics we were working on and what she experienced in school. At first, she said no “not at all.” But then a few minutes later, reflecting on our thinking process during the day, she said:

Anabella: So it's kind of like the overall idea. The bigger idea connects.

Interviewer: There is a bigger idea that connects to the school somehow?

Anabella: Yeah. It's like. It's kinda like those Common Core, you read those Common Core. Have you seen the video where they talk like, they kinda say that the Common Core sucks? And they want to get rid of it. It just makes it hard for kids? But it makes sense why it would be a good thing; because if kids can do this, this way, then they can do anything else.

Here Anabella reflected on how if we can engage children in the high-level mathematical processes similar to those we engaged in during research, for instance problem solving tasks like those promoted by the Common Core, then the students will have the capability to do the lower-level mathematical tasks expected of children in traditional classrooms.

On the fourth day of our collaboration we began by discussing how different 5 cycles can be connected in our graph class; and then we began writing out a proof of our progress thus far. I was writing the proof in what I understood to be the standard format as conventional for mathematicians. But Anabella was not familiar with the conventions of proof writing. She had to ask several clarifying questions to understand the language. Throughout this process Anabella contributed her thoughts on how we should word the proof. At the end of the day, Anabella wrote in her journal:

> Unlike in school there is no right or wrong answer yet. Our math research is 100% based on our knowledge, so we may come up with conjectures that are not true. Human error is our only flaw. This is normal, but it is the reason we may take longer to solve problems.

Through the next few weeks, we had a breakthrough in our work, finding a proof that for graphs in our class with three levels or less, the chromatic number is always three. Yet it took us many days to verify this proof. Anabella was convinced our argument was correct, and yet I needed to mend all the holes that I knew existed in the argument. Anabella learned a lot about the social validation of mathematical knowledge and mathematical communication through this process and throughout the research experience.

As the culmination of our work together, Anabella presented her work twice, both at a local mathematics conference as well as at a student research competition. At those events, Anabella described her big takeaways from the project. She was proud of all the different graphs that she created, inspired by such famous graphs as the Peterson graph and the Grötzsch graph. Anabella said that she learned that “my ideas are important.” She spoke about how she was free to explore her own ideas, and how I (as her research instructor) always took them seriously and considered them. She recognized that as a future teacher, it would be important to “Let mathematics students speak and be heard!” Recognizing how important her own ideas contributed to the research process, she realized that this should carry over to her teaching, as she would need to recognize that her students’ ideas were important and let them express and explore their ideas in the classroom. She also reflected that “communication is key, but the key to communication is knowledge.” She learned about the importance of precise communication of mathematical ideas and the foundational mathematical knowledge necessary to engage in such communication.

Beyond the current study, I have more evidence of what Anabella learned through another study (Pair & Calva, 2020). In that study, students in my transition-to-proof course worked on either the Twin Primes conjecture or the Collatz conjecture, documenting their reflections on the process. Anabella, as part of her pursuit of a supplementary authorization to teach middle school mathematics, was a student in that course. Her reflections showed a sophistication beyond many of the mathematics majors in the course and she was the only student that demonstrated an understanding of the importance of the social validation of mathematical knowledge. Anabella wrote,

> A mathematical proof is a consensual, regarded answer to a mathematical problem. Ideally, it should have no errors (holes) and must convince other mathematicians that it is correct and true for its purpose. [...] If you have a conjecture, the only way that you can safely be sure that it is true, is by presenting a valid mathematical proof. A proof has to be well thought out and tested before being accepted.

At the end of the semester she identified mathematical research as her own and expressed a desire to keep researching: “Although this class has finished, my research has not. I will continue it and the best part is that I will do so at my own pace—wherever I feel like it!”

Conclusion

I presented the results in a story format to paint a picture of Anabella’s research experience, capture what I as the researcher aimed to teach Anabella about the nature of mathematics, and to capture her impressions of mathematical research and what she learned about mathematics and mathematics teaching through the process. Anabella first displayed her skill and interest in mathematics in a content course for preservice elementary and middle school teachers. The current study may be seen as an existence proof that some preservice elementary/middle school teachers are capable of meaningful engagement in research mathematics—and that engaging in research can be a powerful learning experience that can lead to the opportunity to develop their own mathematical identity. Anabella learned about the importance of her own ideas as starting places for mathematical exploration and reflected that it would be valuable for her future students to have similar opportunities. Anabella experienced mathematical communication in a way that also positions her to be a strong mathematics teacher, willing to engage in mathematical discourse with her future students.

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