Not Choosing is Also a Choice

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Combinatorics is a growing topic in mathematics with widespread applications in a variety of fields. It has become increasingly prominent in both K-12 and undergraduate curricula. There is a clear need in mathematics education for studies that address cognitive and pedagogical issues related to students' conceptions of combinatorial ideas. I investigate students' perceptions of the option of not choosing while solving counting problems. In this report, as part of a larger study, I focus on experiences of one undergraduate student who was interviewed in the larger study. The interview was conducted as they solved combinatorial tasks that included the possibility of not choosing a particular attribute when enumerating choices. The data analysis highlighted detecting choices as one of the factors contributing to students' success in solving counting problems. I suggest an extension of a model of students' combinatorial thinking that has been introduced in mathematics education literature.

Keywords: combinatorics, counting problems, detecting choices

Counting problems comprise an essential part of combinatorics and are often used to introduce this area of mathematics. Although counting problems are easy to state there is much evidence that students struggle with understanding and solving them (e.g., Eizenberg & Zaslavsky, 2003; Godino, Batanero, & Roa, 2005, Batanero, Navarro-Pelayo, & Godino, 1997). The existence of abundance evidence might be one of the reasons that draw the field's attention to the importance of combinatorics in mathematics curricula (e.g., Lockwood, Wasserman & Tillema 2020). As counting problems are currently part of K-12 and undergraduate curricula, there is a necessity to study factors that might have affected students' success. To this end, addressing students' ways of thinking at a level that benefits a deeper understanding of how students conceptualize counting problems is significantly essential. Lockwood (2013) conceptualized students' combinatorial thinking by considering students' ways of thinking about counting problems based on three components—formulas/expressions, counting processes, and sets of outcomes. In solving a counting problem, errors may come from not correctly detecting options/choices which the problem specifies. In fact, detecting all the choices in solving a counting problem is one of the factors of students' success. However, options/choices can be presented in different ways. In this study, I narrow the focus to students' understanding of the particular option of not choosing as a choice and how students may consider it in their counting activity.

Theoretical Framework

Lockwood (2013) presented a theoretical framework that provides a model of students' combinatorial thinking. The focus of this model was on the ways students conceptualize counting problems. In fact, this model facilitates a conceptual analysis of students' thinking in facing counting problems and it provides a language to address and describe different aspects of students' activities related to combinatorial enumeration (counting). It has equipped educators with ways of elaborating on how students might think in solving counting problems, and it has provided a powerful lens for analyzing different facets of students' counting activity. This framework has three components, formulas/expressions, counting processes, and sets of outcomes. Any mathematical expressions that results some numerical values can be considered as a formula/expression. While solving a counting problem, a counter may engage in an enumeration process or series of processes. The counter either performs some steps or procedures or even imagines doing them. Both cases are considered as enumeration processes which describe counting processes as a component in this model. Sets of outcomes refer to the sets of elements that a counter wants to count. Any collection of objects that the counter tries to generate or calculate by a counting process, is considered as a set of outcomes. The relationship among the three components of this model is illustrated in Figure 1.

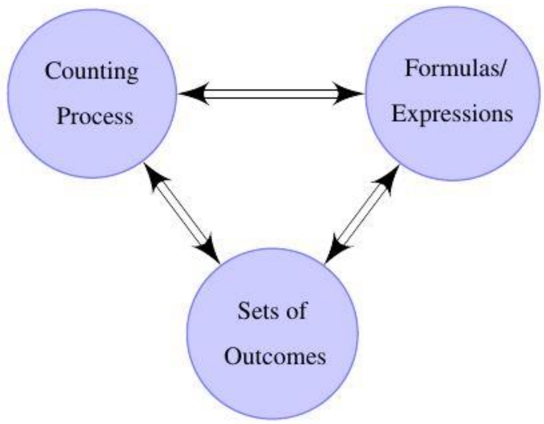

Figure 1: Lockwood's model of students' combinatorial thinking

Methods

In this study, as part of a larger study, I report to an interview with one of the students who was involved in the study that focused on students' perception of choosing. The participant, Harry, was an undergraduate student who attended a discrete mathematics course (Discrete mathematics II). The course focused on counting problems, graph theory, trees, inclusion-exclusion, generating functions, recurrence relations, and optimization and matching. The student participated in an individual semi-structured interview, one hour long. The interview instrument consisted of two problems that aimed at probing students' understanding in how they consider the possibility of not choosing and comparing students' understanding of "choosing" to "not choosing". I designed the two problems with the goal of highlighting students' judgments in validating the possibility of not choosing as a choice.

Problem 1: There are 9 international students and 5 domestic students to fulfill English language requirement. International students have to take one of three English courses (ENGL 100A, ENGL 100B, ENGL 100C). There is no requirement for domestic students to take any of these three courses. Taking one of these three courses is an option for them. How many possibilities are there for students' choices. (for example, one possibility is 3 international students take ENGL100A, 3 international students take ENGL 100B, 3 international students take ENGL 100C and all domestic students take ENGL 100B)

Problem 2: A group of 8 people had a meeting. Three of them had a presentation in this meeting. After their presentations they desperately needed a cup of coffee. The people who did not present can choose to have coffee. There were 11 different types of coffee to choose from. How many possibilities are there for these people's choices? Amy wrote the following answer:

$$\binom{5}{0}\times 11^3+\binom{5}{1}\times 11\times 11^3+\binom{5}{2}\times 11^2\times 11^3+\binom{5}{3}\times 11^3\times 11^3+\binom{5}{4}\times 11^4\times 11^3+\binom{5}{5}\times 11^5\times 11^3$$

Do you agree with her? What was she thinking about? Nikki's answer was $11^3\times 12^5$. What do you think about Nikki's answer?

The problems were designed such that a possible solution—$3^9\times 4^5$ for Problem 1, and $11^3\times 12^5$ for Problem 2—includes consideration of not choosing a particular attribute. The option of not choosing is present in different formations in each problem.

Results

Although the participant's work shows he is familiar with the fundamental principles of counting, his solutions indicate he was consistent in not considering the option of not choosing in the two problems. Although different ways of presenting this option did not affect his consideration of the option of not choosing, it affected his combinatorial thinking.

Organizing the Data by the Three Components of Lockwood's Model

After highlighting noteworthy parts of the participants' responses, I categorized them into the three components of Lockwood's model. I have assigned each highlighted situation to the components of Lockwood's model that have played an important role in making the situation significant. Pushing students to initiate their counting activity by one of the three components does not necessarily keep the rest of their counting activity in the same component.

The option of not choosing and formulas/expressions. The two numerical expressions in the second problem pushed Harry to start his combinatorial thinking with the formulas/expressions component. To be able to evaluate $11^3\times 12^5$ as an answer, the extra option of not having a coffee needed to be numerically explained. Harry started working on the second problem by making a connection between the addition principle and the plus signs in Amy's answer.

Harry: So, what she has is that there is eleven cubed that's constant among each of her expressions and it looks like she also separated it by case by case basis which you can tell by rule of sum from the plus signs.

Harry perfectly interpreted Amy's counting thought and precisely made sense of each counting step he had defined based on Amy's expression. His consideration of Nikki's answer started by interpreting 11^3 correctly and continued as follows:

Harry: For the five optional people the ones who don't have to have coffee she gave each of them twelve choices instead of eleven for the coffee, so what I believe she did was, we know that there are eleven types of coffee, for each person, she added one more choice, which is whether they are going to have coffee or not. So, say this is person number one here, they have eleven different types of coffee to choose from if they choose coffee, she thought of the option where that person does not choose coffee as an option. So, it goes from eleven to twelve options.

He continued by writing out 5 dashes, writing 12 in each of the dashes. He indicated each 12 refers to the 11+1 options. Although $11^3\times 12^5$ brought up the idea of considering the option of not choosing as an option, supportive counting processes were needed to convince Harry. I discuss Harry's hesitation in acknowledging the option of not choosing in the following subsection.

The option of not choosing and counting processes. Although multiplication and addition principles which are the two foundations of counting processes presented in Harry's counting activity, detecting the option of not choosing appeared problematic. After Harry explained what might be going on in Nikki's answer by clarifying that 12 in her answer comes from 11+1, I asked him if he agreed with her and he answered as follows:

Harry: No. then I am also biased because this is [pointing to Amy's answer] exactly how I approached the problem the first time and it's just the way I think of it when I like splitting up the cases deciding by rule of sum and the rule of product to determine the stages, that's just how I think. But then hers logically doesn't seem fundamentally sound because it seems even though she umm. I don't know! It just doesn't feel right.

Although He initially felt it may make sense to consider the option of not choosing as a choice, he was not convinced to validate this possibility as a choice. He explicitly mentioned, "it just doesn't feel right". I suspect that he had not seen/considered this kind of choice in his previous counting activity and/or in his routine life. After I asked him to explain why he thinks considering the twelfth flavor as "no coffee" is not right, he answered,

Harry: for twelfth option she decided to have no coffee. I am trying to see if, um, I think the thing that might be confusing here is that there is no rule of sum differentiating the cases. So, the way I have been approaching problems and the way I have seen the most of the answers usually is if there are cases involved there is always the rule of sum that separating them. What is giving me confusion here is that we only have a rule of product. Yes, she has added an extra option to account for there being no coffee chosen, but intuitively it feels wrong to me. If that makes sense. I can't put words to it but if that speaks about my own understanding if I can explain why it is not correct. I feel it is not correct, but I cannot give an argument as to why she isn't right. Because I totally understand where she is approaching the problem from.

Harry could interpret Nikki's expression, but it seemed he was confused if Nikki's answer was correct. Harry's discussion points to "met-before" construct introduced by Tall (2004). The notion of met-before concerns how prior knowledge and previous experiences affect learning in a new context. A "met-before" is a personal mental structure in our brain as a result of experiences met before and can act both positively and negatively when a new concept is introduced (Tall, 2004; 2008). Harry's previous experience seems to build a connection between the existence of cases in counting problems and the necessity of applying the addition principle to solve the problem. This connection is an obstacle for considering any other approach that does not attend to the cases. In fact, it seems his met-before experience affects negatively on accepting the option of not choosing as a choice. Interestingly, to evaluate Nikki's answer, he tried to overcome this obstacle by justifying that multiplication can be seen as repeated addition, so having just multiplication in Nikki's expression does not necessarily make the answer wrong. However, the equality of these two suggested answers (expressions) come from the equality of their sets of outcomes (based on proper choices and counting processes), and not a specific manipulation that rewrites repeated additions as multiplication. I suggest that not validating the possibility of not choosing as a choice with supporting counting processes directed Harry's attention to expressions.

He began working on the first problem by reading it and underlining some words and phrases such as "9 international", "5 domestic", "one of three", "no requirement", "one of these three courses", and "option". Although he highlighted the keywords in this problem, his first interpretation of the problem was incorrect as follows:

Harry: So, it is possible for me to just only distribute the nine international students among the three courses but do nothing with the five domestic students.

In solving a problem, having an intended interpretation of what the problem asks is essential. Particularly in counting problems, detecting all the options/choices correctly would be the main part of interpreting the problems. Harry's interpretation of having the possibility of not choosing any English course for domestic students made him think he does not need to make them involved in his counting activity. The option of not choosing has been missed in his first consideration. After he correctly counted all the possible ways for international students' choices and reached 3^9, I asked him to think about domestic students. Although he initially eliminated domestic students from his counting activity, he started thinking about them by making an interesting distinction between the two principles. He made the distinction to explain where to apply the two principles as follows:

Harry: What I remember in MACM101 [Discrete mathematics I] we learned the shortcut was that, one shortcut was when you are doing anything involving "and", like saying student one takes this and student two takes that, is the rule of product, but if you have "or", either student one takes this, or student two takes this, you do the rule of sum.

His explanation about the two principles points to what Tucker (2002) mentioned, "Remember that the addition principle requires disjoint sets of objects and the multiplication principle requires that the procedure break into ordered stages and that the composite outcomes be distinct" (p. 170). Harry's statement highlights two correspondences between, 1) "and" and the multiplication principle and 2) "or" and the addition principle. His language seems to suggest he was looking for keywords in distinguishing the proper operation in his solution. However, he went on to articulate why considering different cases would be legitimate as follows:

Harry: this combination or number of possibilities [pointing to his solution that was 3^9] will stay the same for the international students, regardless whatever the domestic students are doing. So, it does not have any effect on the international students at all. So that's ok if we try cases.

He began to list the cases systematically according to how many domestic students get an English course. In addition to the interesting distinction he made between the two principles I discussed above, he made the second distinction as follows:

Harry: My intuition is saying that in this case [only one domestic student takes a course] it is still the inside the case and not across the cases and inside the cases is still the rule of product, because with one domestic student taking a course really you are adding a tenth stage. The international students they have their nine stages of choosing, now you add tenth stage and it would be three to the ten, because now there is three more choices, how will that domestic student choose out of the three courses. I do the rule of sum across the cases, because this is where it is like either you have the situation where no domestic students take any course, or one student takes it or two domestic, or three or four.

When he talked about "cases" above, his language of "inside the case" and "across the cases" indicate that he is aware of how to use the addition and multiplication principles. Although he made some insightful comments about how he decided to apply the two principles, his language suggests some keywords for each principle. It seems not validating the possibility of not choosing as a choice makes him consider several cases for domestic students and use the addition principle. Harry continued by trying to make Problem 1 fit into a combination with repetition model, usually called the "stars and bars" technique. He applied the model to count all the possible ways for both international and domestic students identically. It seemed he applied one formula for both international and domestic students regardless of the possibility of not choosing to take an English course for domestic students. After I asked him what the difference is between international students and domestic students, he read the problem several times and ended up crossing his answer. He was ultimately not able to realize the option of not choosing can be one option for domestic students and the problem can be solved without considering different cases.

The option of not choosing and sets of outcomes. While Harry initially was not assured that the option of not having a coffee in the second problem can be considered as the twelfth flavor of coffee, his attempt to consider one possible case was convincing for him.

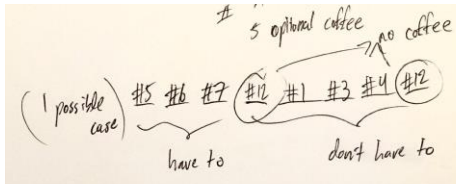

Figure 2: Harry's example of an element of the set of outcomes

Harry: When I think about it and I break it down this way [pointing to the first three dashes in his solution] this is just one case, here I add the other part as well [adding five more dashes]. These are the three people that they have to. These are the five people who don't have to. And we say this person have coffee number five, coffee flavor six, coffee flavor seven. This is one possible way that everything is distributed. If these two people no coffee, it is the same. I believe it is logically valid by adding the twelfth choice. And that choice is no coffee.

Harry's explanation shows his awareness of the overall conditions of the problem. Interestingly, considering one element of the set of outcomes helped him to convince himself that the option of not choosing is an option. In fact, the case he examined facilitated his insight into picturing the option of not choosing in the setting of the problem to validate this type of option. I continued the interview as follows:

Interviewer: So, you also agree with Nikki?

Harry: Yeah!

Interviewer: You agree with both?

Harry: But I usually approach problems like this [pointing to Amy's answer] and this is the only way I have seen it, I agree with Amy's answer more. I am much more hundred percent about Amy's but when I break Nikki's option dawn into one case, I can see that should be nothing wrong with adding the twelfth option.

Interviewer: So, you are not hundred percent agree with Nikki?

Harry: Yeah! Let's go with that. I can say, I agree with Amy hundred percent, Nikki I can see where her logic coming from but since I cannot fully endorse Nikki's answer. I understand it, I see what she did, I can break it dawn. But I don't understand the problem to the point where I can say confidentially that she is right. So, I go with Amy.

Although he reached the point to acknowledge the option of not choosing as an option, due to lack of supportive counting processes, he could not make himself convinced with Nikki's answer.

Discussion and Conclusion

Although detecting the option of not choosing is not a counting process, it definitely affects the process of counting. This research confirms rejecting the option of not choosing as an option may cause a detour in students' paths of solving counting problems by increasing the number of processes. In addition, not having it as an option may force counters to go through different structures of counting processes. Both scenarios make counting processes become more complicated. In addition, while detecting the option of not choosing cannot be considered as any combinatorial thought related to the other two components of the model, not detecting it as an option may affect both. Indeed, detecting the option of not choosing belongs to a broader aspect of students' combinatorial thinking which I call *detecting choices*. My data shows by including this aspect of students' combinatorial thinking the model will benefit from a refinement. Detecting choices would be a necessity for any counting process. However, while in Lockwood's model detecting choices appears to be part of counting processes, my data brings the conclusion that detecting choices is a separate element connected to all the three components in Lockwood's model.

In solving a counting problem, the numbers of choices that the problem specifies are needed to be detected in order to make any enumeration process (counting process) begun. The data shows detecting choices is not part of counting processes but it is definitely needed in counting processes. Lockwood implicitly considered students' combinatorial thinking related to choice detection as part of counting processes. However, the data demonstrates detecting choices carries an important role in students' combinatorial thinking that makes it deserved to have more attention. It appears merging the two important aspects of students' combinatorial thinking—detecting choices and counting processes—decreases the facilities this model could have brought for educators and teachers in this field. In fact, Lockwood's model does not equip researchers to determine how students might detect choices in their counting activities while their combinatorial thinking does not involve in any counting process. This is important as it demonstrates a need to refine the framework to include this aspect of students' combinatorial thinking.

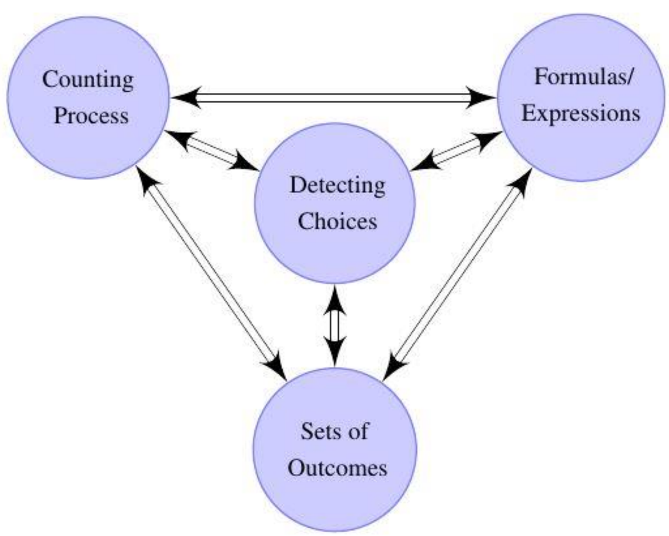

Figure 3: An extension for Lockwood's model of students' combinatorial thinking

The data shows detecting choices tremendously affects the other three components. Hence, detecting options/choices not only could be part of students' combinatorial thinking but also could be the core of students' counting activity. So, I locate the fourth component in the core of Lockwood's model.

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