Relative Size Reasoning: A Cross-Cutting Way of Thinking for Learning Precalculus Ideas

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The goal of this paper is to describe a cross-cutting way of thinking which may be productive for learning precalculus ideas. I describe the construct, "relative size reasoning" which consists of thinking about the relative size of two quantities as they change together. This way of thinking entails quantitative and covariational reasoning. I describe the mental operations involved in relative size reasoning and how it can be used by providing three illustrations of relative size reasoning in three precalculus ideas and conclude with future research.

Keywords: relative size reasoning, relative size, way of thinking, quantitative reasoning

Introduction

The ideas of measurement and measurement comparisons (such as fractions or ratios) are mathematical concepts introduced to students in elementary school. However, students in university mathematics courses struggle with comparing two *quantities* in terms of their *relative size (e.g Byerley, 2019; Tallman, 2015)*. Thinking about the *relative size* of two quantities entails envisioning the measure of one quantity in units of the other, provided the two quantities are of the same unit of measurement (the magnitude of one measured in units of the magnitude of the other). If they are not of the same unit of measurement, it involves thinking of the measure of one quantity in multiples of the measure of the other quantity. For this paper, I introduce the idea of relative size reasoning to characterize the dynamic imagery involved in thinking about the relative size of two quantities as the two quantities' values change in tandem (Thompson, 1994; Carlson et al., 2002).

Traditional or conventional mathematics textbook designers create sections and chapters that are disconnected and developed without regard to NCTM standards for mathematics teaching and learning (Cady et al., 2015). Teachers rely heavily on these disconnected textbooks and curriculum to structure their classroom instruction (Valverde, Bianchi, Wolfe, Schmidt, & Houng, 2002). As a result, teachers are not equipped to support students in reasoning about these connections in the classroom (Thompson & Saldanha, 2003). Thus, students are not supported in developing the cross-cutting "way of thinking" that is productive for understanding foundational ideas (e.g., rate of change). For example, students who do not view a quotient as representing the relative size of the ratio's numerator and denominator will not be equipped to understand the idea of constant rate of change.

The foundational reasoning necessary for thinking about constant rate of change includes *quantitative reasoning* (Thompson, 1990, 2011), reasoning multiplicatively (Thompson 1994b; Thompson and Thompson 1994, 1996; Thompson & Carlson 2017), coordinating two covarying quantities and their changes (*covariational reasoning*) (Thompson & Saldanha, 1998; Carlson et al., 2002; Thompson & Carlson 2017), and thinking about the proportional correspondence between the changes in the two quantities. Relative size reasoning is useful for thinking about constant rate of change because it provides students with an opportunity to approach problems with a way of thinking that increases their ability to make meaning for the mathematical structures they create, and determine the appropriate operations needed to evaluate some quantity in the problem, which entails *quantitative* and *covariational reasoning*. Lobato and colleagues have reported that many students do not view the idea of proportionality, slope, rate of change, and average rate of change as being connected (Lobato, 2006; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002). In fact, it has been documented that university students have difficulty modeling function relationships involving rates of change (Carlson, 1998; Thompson 1994a). Specifically, understanding ideas such as constant rate of change is central to understanding ideas in calculus such as (a) the difference quotient (Byerley & Thompson, 2014), (b) the concept of the derivative (Byerley et al, 2012; Zandieh, 2000) and (c) the fundamental theorem of calculus (Thompson 1994). These findings suggest that students may need to be supported in using relative size reasoning. Comparing two quantities multiplicatively has been documented to be important for understanding ideas in specific mathematical topics such as measurement (Thompson et al., 2014), constant rate of change (Thompson, 1994a, 1994b, 1995; Thompson & Saldanha, 1998; Carlson et al. 2002), rational functions (Pampel, 2017), trigonometry (Moore, 2014, 2010; Tallman & Frank, 2018; Thompson, 2008), exponential functions (O'Bryan, 2018), and difference quotient (Byerley, 2019). More research is needed to understand the nature of relative size reasoning, how it develops in students, and how it contributes to learning other ideas in mathematics. Since *relative size reasoning* has not been an explicit focus of research, this study investigated the question:

What are the mental operations involved in relative size reasoning and how is relative size reasoning used?

Review of the Literature

I initially discuss how relative size reasoning is used when learning and understanding precalculus level ideas. I follow by providing an overview of prior research that discussed the role of relative size reasoning when learning topics in elementary school through calculus.

Relative Size Reasoning in Rational Functions Literature

When one has reasoned about rational functions productively, she can "coordinate the covarying relationship of the polynomial in the numerator (the input and output quantity of the polynomial) and the polynomial in the denominator (the input and output quantity of the polynomial)" (Pampel, 2017). This relationship represents the relative size of the value of the numerator in terms of the value of the denominator, which also varies with the input quantity. Using relative size reasoning to think about rational functions leads to reasoning quantitatively about characteristics of rational functions like end behavior, asymptotes, and holes rather than relying on algebraic manipulations (Pampel, 2017).

Relative Size Reasoning in Trigonometry Literature

Thompson (2008) argued that students' difficulties in understanding trigonometry lie in the essential meanings they construct from grades 5 through 10, such as their meaning for an angle. Students are frequently introduced to the idea of angle measure as representing a location of an angle's rays (e.g., a right angle is 90 degrees, and a straight angle is 180 degrees). It is also common for students to have weak or incoherent meanings for what it means to measure an angle's openness in degrees or radians. Measuring an angle's openness in degrees consists of thinking about the angle's openness in units of $1/360^{\text{th}}$ of any circle's circumference. When measuring an angle in radians, we compare the size of the arc length relative to the circle's radius length (Moore, 2014, 2010; Tallman & Frank, 2018; Tallman 2015; Thompson, 2008). In other words, we are measuring the subtended arc of any circle cut off by the angle's rays in units of its respective radius length.

Relative Size Reasoning in Exponential functions Literature

O'Bryan (2018) claimed that understanding the idea of exponential functions relies on comparing two quantities multiplicatively. When one has reasoned about exponential functions coherently, she can "conceptualize an exponential function as a function with a constant growth factor or constant percent change over all equal sized intervals of the domain" (O'Bryan, 2018). In other words, students with a more developed understanding of exponential functions recognize that the growth factor for an n-unit change in the independent variable is a comparison of the relative size of two function values over that n-unit change.

Relative Size Reasoning in Division and Fractions Literature

Thompson and Saldanha (2003) characterized a mature understanding of fractions as involving multiplicative reasoning and strong meanings for measurement, multiplication, division and quotient. When students understand the idea of division (which entails measuring and partitioning) they understand that "any measure of a quantity induces a partition of it and that any partition of a quantity induces a measure of it" (Thompson & Saldanha, 2003). When fractions and division are thought of multiplicatively, "A is m/n times as large as B" means that A is m times as large as $1/n$ of B (Thompson & Saldanha, 2003; Byerley & Thompson, 2012; Thompson et al., 2014). Thompson and Saldanha (2003) emphasize a direct link to reciprocal relationships of relative size to conceptualize fractions. This is similar to a person who is thinking with a *Steffe Magnitude* which Thompson, Carlson, Byerley, & Hatfield (2014) defined as one who is reasoning about measurement of a quantity's size relative to the size of its unit and the reciprocal relationship between them. Thompson et al. (2014) described a conceptual analysis for developing a mature magnitude scheme that involves ideas such as multiplication, division, proportional correspondence, measure, fraction, and quotient as a measure of the relative size of two quantities. Conceptualizing division as relative size enables students' conceptualization of divisors that are not-integer values (Byerley, Hatfield, and Thompson, 2012). Understanding reciprocal relationships of relative size entails conceptualizing the connectedness between ideas of measure, multiplication, fractions, and division. Thompson and Carlson (2017) and Byerley, Hatfield, and Thompson (2012) also argued that to understand the concept of rate of change, it involves an understanding of quotient, ratio, accumulation, and proportionality, which consists of thinking about the comparison of two quantities.

Theoretical Perspective

The theoretical perspective used for this study is radical constructivism (Glasersfeld, 1995) which proposes that every individual has their own reality, knowledge is constructed based on previous personal experiences, and knowledge lies in the mind of the individual. It is important to note that knowledge is not transferable from one person to another. Only through models can we portray hypothetical models of students' *schemes* and *meanings*. A scheme is an "organization of actions, operations, images, or schemes which can have many entry points that trigger action – and anticipations of outcomes of the organization's activity" (Thompson et al., 2014). When I say meaning I am referring to the "ideas or ways of thinking that someone intends to convey to someone else and uses signs or symbols to do so" (Thompson, 2013). Meanings that a person has exist in the mind of that individual and the person that is interpreting them. Therefore, my intention is not to classify how all students engage in relative size reasoning, but rather provide a model of an epistemic student's thinking (Steffe & Thompson, 2000).

Relative size reasoning is a *way of thinking* that is needed for conceptualizing a wide range of mathematical ideas, while engaging in relative size reasoning relies on one having mature understandings of measurement, fractions, multiplication, division, ratio, proportionality, and magnitude. Relative size reasoning includes comparing the size of two quantities by envisioning the measure of one quantity in units of the other, provided the two quantities are of the same unit of measurement (the magnitude of one measured in units of the magnitude of the other). If they are not of the same unit of measurement, it involves thinking of the measure of one quantity in multiples of the measure of the other quantity. When I refer to a *way of thinking*, I am referring to a pattern that a person developed for utilizing specific meanings for reasoning about specific ideas (Thompson et al., 2014). Constructing productive meanings for these ideas involves *quantitative reasoning* (Thompson, 2011) and *covariational reasoning* (Carlson et al., 2002; Thompson and Carlson, 2017).

To engage in relative size reasoning, one must first engage in *quantitative reasoning* (Thompson, 1990, 1993, 1994, 2011) which begins with conceiving of a quantity. A *quantity* is a "quality of something that one has conceived as admitting some measurement process" (Thompson, 1990). *Quantification* is the process of assigning numerical values to quantities (Thompson, 1990, 2011). Thompson (1990, 1994b, 2011) described *quantitative reasoning* as "the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships." *Quantitative relationships* are 3 quantities where two quantities and a quantitative operation are used to make a third (Thompson, 1990). For instance, constant speed is a quantity that determines how many times as large the change in distance traveled is, in comparison to the corresponding change in time. In order to make this comparison, the student must also conceptualize the change in distance and change in time as quantities. Then the student must think about the changes in the quantities as the quantities vary in tandem which entails *Covariational Reasoning*, "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other." (Carlson et al, 2002). Thompson and Carlson (2017) argued that covariational reasoning entails developing the ability to imagine quantities' values varying smoothly and continuously.

Three Vignettes of Relative Size Reasoning

I propose that relative size reasoning is a cross-cutting *way of thinking* that is used when learning and using numerous ideas in mathematics. I provide three vignettes grounded in research previously presented above to illustrate a student's mental operations when engaging in relative size reasoning and why it is propitious for conceptualizing these situations.

Vignette 1

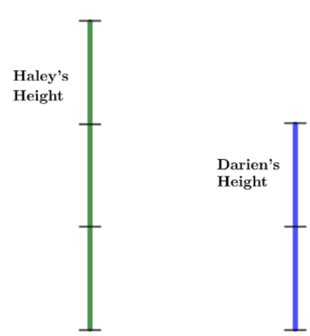

Figure 1. An image to represent the comparison of Haley and Darien's Heights

Suppose a student is first asked to compare Haley's height to Darien's height and then Darien's height to Haley's height (see Figure 1). The student must conceptualize the quantities in the situation: Darien and Haley's heights. When the student compares Haley's height to Darien's height, she must imagine measuring the magnitude (length) of Haley's height relative to the magnitude (length) of Darien's height. So the student must conceptualize Darien's height as the unit of measure. Therefore, using her measurement scheme, she imagines Darien's height as a measuring stick to determine Haley's height in units of Darien's height. Then using her scheme for fraction and division, she may think about how many times as tall Haley's height is relative to Darien's height. A student who exhibits these mental actions likely understands and can express that Haley is $\frac{3}{2}$ times as tall as Darien.

Then when comparing Darien's height to Haley's height, the student must imagine measuring the magnitude (length) of Darien's height in units of the magnitude (length) of Haley's height. Now she must reconceptualize the situation where Haley's height is the measuring stick. Then the student may think about the multiplicative comparison as Darien is $\frac{2}{3}$ times as tall as Haley. Engaging in relative size reasoning to compare Haley and Darien's heights entails thinking of each measured height as a measurement unit or measuring stick used to measure the other person's height.

An example of a student who reasons in a way other than using relative size reasoning to think about comparing Haley and Darien's heights may think about the fraction $\frac{2}{3}$ or $\frac{3}{2}$ as "2 over 3" or as a rule that tells you to divide 2 by 3. This may result in the student seeing a fraction as a command to divide, and less likely to conceptualize the result of division as determining the relative size of one quantity in terms of another.

Vignette 2

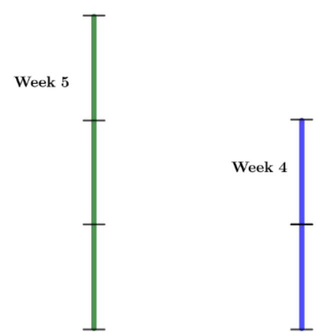

Figure 2. An image of the sunflower's height from week 4 to week 5. Task adapted from CCR exam (Carlson, Madison, West, 2015).

Suppose we consider a task adapted from a CCR exam (Carlson, Madison, & West (2015). A sunflower is planted and grows by 50% each week as pictured in Figure 2. Given the height of the sunflower after 4 weeks, a student must find how tall the plant will be in week 5. Engaging in relative size reasoning while responding to this problem entails imagining the quantities in the problem context: the time in weeks since the sunflower was planted and the height of the plant. The student must visualize both the time since the sunflower was planted and the height of the sunflower as measurable attributes of the plant and consider the height of the sunflower while imagining the amount of time elapsed since the sunflower was planted.

When thinking about this situation the student must have a meaning for percent which entails understanding that the size of Quantity A is n% of the size of Quantity B means that the size of Quantity A is n one one-hundredths of the size of Quantity B (O'Bryan, 2018). The student's percent scheme includes the idea that the quotient $\frac{A}{B} = n \cdot 0.01$; the student interprets these symbols as expressing that the size of Quantity A is 0.01 times as large as Quantity B. Therefore, she sees that Quantity B is the measuring stick used to determine the size of Quantity A. In addition, the student must have a meaning for percent change that entails describing the change in the value of a quantity as a percentage of the original value.

Specifically, for this question, a student must imagine using the height of the plant in week 4 as the measuring unit. Given that the plant grows by 50% each week, one must conceptualize the sunflower growing by 50% and seeing an increase of 50% as the percent change as each week passes. So, when comparing the height of the sunflower in week 5 to week 4, she would see that the height of the sunflower in week 5 is 150% of the sunflower's height in week 4. The student would see the multiplicative comparison of the height of the sunflower in week 5 as 1.5 $or\ \left(\frac{3}{2}\right)$ times as tall as it was in week 4. Determining the reciprocal relationship of the height of the sunflower in week 4 relative to week 5 requires conceptualizing the sunflower's height in week 5 as the measuring unit. Understanding the reciprocal relationship in terms of percentage entails the idea that the height of the sunflower in week 5 is the reference quantity and the height in week 4 is about 66.67% of the plant's height in week 5. Thus, the sunflower's height in week 4 is $\frac{2}{3}$ times as tall as it is in week 5. Engaging in relative size reasoning to compare the heights of the sunflower for weeks 4 and 5 entails thinking of the sunflower's height in a certain week as a measurement unit used to measure the sunflower's height for another particular week.

Vignette 3

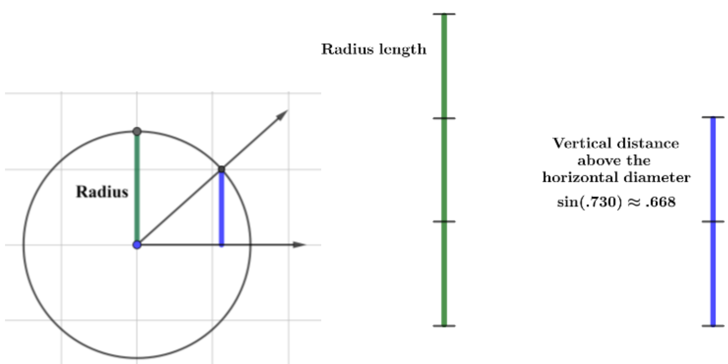

Figure 3. Left: An image of a circle with an angle centered at the circle's center with a terminal point at $sin(0.73)$*. Right: An image of the height of the terminal point's vertical distance above the circle's center relative to the radius of the circle.*

Another example of using relative size reasoning to make sense of a situation would be to describe the meaning of the output value of the sine function given an angle measure of 0.73 radians as pictured in Figure 3. Using relative size reasoning to think about the sine function entails imagining that, when given an angle whose initial ray is in the 3 o'clock position and vertex is at the center of a circle, the sine function will take the angle's radian measure as an input and output the terminal point's vertical distance above the circle's center measured in radius lengths. The student would have to conceptualize the radius length as the measuring stick. With this meaning the student may reason about the terminal point's vertical distance above the circle's center in radius lengths given an angle with a measure of 0.73 radians using the sine function: $sin(0.73) \approx 0.668 \approx \frac{2}{3}$. Thus, she must conceptualize the relative size of the terminal point's vertical distance above the circle's center measured in units of the radius lengths. The student must conceptualize that the terminal point's vertical distance above the circle's center measured in radius lengths is $\frac{2}{3}$ times as long as the radius length. Understanding the reciprocal relationship entails using the terminal point's vertical distance above the circle's center as the unit of measure for determining the relative size of the radius in comparison to the terminal point's vertical distance above the circle's center. This conceptualization includes seeing that the radius length is $\frac{3}{2}$ times as long as the terminal point's vertical distance above the circle's center.

An example of a student who views the meaning of the sine function in ways other than relative size reasoning may include thinking about "opposite over hypotenuse" or "SOH" in the acronym SOH-CAH-TOA. Students who learn trigonometric functions as SOH-CAH-TOA will be at a disadvantage when it comes to needing to think about the arguments of these functions and what they represent. Especially since recognizing that angle measure in radians represents this proportional relationship between the subtended arc of any circle cut off by the angle's rays and the corresponding circle's radius requires relative size reasoning. Engaging in relative size reasoning to compare the terminal point's vertical distance above the circle's center to the circle's radius entails thinking of each measured length (either the radius or vertical distance) as a measurement unit or measuring stick used to measure the other length.

Discussion, Limitations & Future Research

Since conjecturing *relative size reasoning* is foundational for understanding many ideas in mathematics, it is necessary to emphasize this as an important way of thinking for students to acquire. When engaging in relative size reasoning, thinking about comparing any two quantities (fixed or varying) in the previous situations, one must reason about the relative size of one quantity in units of the other and see the reciprocal relationship as well. Students who think about fractions and division as a command to divide may be limited to seeing the meaning of the result of division and the connections between their ways of thinking while approaching problems in different topics. Students who think about trigonometric functions as an acronym may be limited when it comes to thinking about measuring an angle in radians. Empirical studies are needed to further refine the conceptual analysis presented, to further investigate how students spontaneously use (or don't use) relative size reasoning, and to explore the role of relative size reasoning in learning and using other ideas in precalculus and calculus.

I hypothesize that supporting student's in engaging in relative size reasoning will strengthen their ability to characterize the covariation in two quantities as their values change together and understand key concepts (constant rate of change, exponential growth, derivative) in mathematics.

Future Research Questions:

1. What meanings do students display or exhibit about the concept of relative size across multiple precalculus topics: constant rate of change, proportionality, exponential functions, rational functions, and trigonometry?
2. What are ways that a mathematics curriculum and a teacher can support students in developing relative size reasoning?

References

Byerley, C., Hatfield, N., Thompson, P. W. (2012). Calculus students' understandings of division and rate. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.) *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 358-363). Portland, OR: SIGMAA/RUME.

Byerley, C., & Thompson, P. W. (2014). Secondary teachers' relative size schemes. In P. Liljedahl & C. C. Nicol (Eds.), *Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 217-224). Vancouver, BC: PME. Retrieved from http://bit.ly/1qyqmaK.

Byerley, C., & Thompson, P. W. (2017). Secondary teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior*, 48, 168-193.

Cady, J., Hodges, T., & Collins, R. (2015). A Comparison of Textbooks' Presentation of Fractions. *School Science and Mathematics*, *115*. https://doi.org/10.1111/ssm.12108

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education III* (pp. 114-162). Providence, RI: American Mathematical Society.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while

modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33,* 352–378.

Carlson, M., Madison, B., West, R. (2015). The calculus concept readiness exam: Assessing student readiness for calculus. In K. Marrongelle; C. Rasmussen; M.O.J. Thomas (Eds.) International Journal of Research in Undergraduate Mathematics Education, 1-25, DOI 10.1007/s40753-015-0013-y

Moore, K. C. (2010). The role of quantitative reasoning in precalculus students learning central concepts of trigonometry. Unpublished Ph. D. dissertation, Arizona State University, Tempe, AZ.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. Journal for Research in Mathematics Education, 45(1), 102-138

Moore, Kevin & Laforest, Kevin. (2014). The Circle Approach to Trigonometry. Mathematics Teacher. 107. 616-623. 10.5951/mathteacher.107.8.0616.

O'Bryan, A. E. (2018a). Exponential growth and online learning environments: Designing for and studying the development of student meanings in online courses. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University

Pampel, K. (2017). *Perturbing Practices: A Case Study of the Effects of Virtual Manipulatives as Novel Didactic Objects on Rational Function Instruction*. Arizona State University.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In *North Carolina State University*.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Tallman, M. (2015). An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.