Foundational Ideas for Understanding the Constant Rate of Change

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This report presents a conceptual analysis that describes the productive ways of thinking of a hypothetical student in learning the idea of constant rate of change. The paper characterizes a hypothetical learning trajectory with a hierarchy of ideas that we conjecture to be foundational for understanding constant rate of change. The study briefly presents an instructional sequence of tasks guided by the hypothetical learning trajectory to promote learning the foundational ideas for understanding the constant rate of change.

Keywords: constant rate of change, conceptual analysis, hypothetical learning trajectory, quantitative reasoning, proportional reasoning

Introduction

Across the country, most Precalculus curriculum in college repeats the procedural approaches that students learn in high school, and therefore, students are less successful in passing Calculus in college (Sonnert & Sadler, 2014). Byerly (2016) reported Calculus students' and teachers' weak meanings of measurement, fraction, and constant rate of change. Students start building their meanings for constant rate of change and proportionality in their early mathematics learning. In Precalculus and Calculus, students get the opportunity to build meanings for the idea of constant rate of change in a sophisticated way. Thompson (2008) discusses that the constant rate of change is foundational to understanding linear functions and supporting ideas like an average rate of change, proportionality, and slope. Studies have reported the disconnections students and teachers have as they treat these ideas as different sets of actions and as associated with unrelated contexts (Lobato, 2006; Lobato & Siebart, 2002; Lobato & Thanheiser, 2002; & Coe, 2007). The coherent meanings for ratio, rate, and proportionality are foundational to the idea of constant rate of change.

My goal is to study an instructional sequence that is designed to support students in understanding and connecting ideas of a quantity, change in a quantity, variable, formula, ratio, rate, linear function, slope, an average rate of change, etc. that are related to learning and understanding the idea of a constant rate of change. Reasoning abilities that support students in making connections between the ideas related to a constant rate of change involve quantitative reasoning (Thompson, 1994), covariational reasoning (Carlson, Jacobs, Coe, Larsen, Hsu, 2002; Thompson & Carlson, 2017), and proportional reasoning (Lamon, 2007; Lesh, Post & Behr, 1988; Thompson, 1994). If we say two quantities have a constant rate of change, then we mean that two quantities are varying together; any change in one quantity’s variation remains proportional to the corresponding change in another quantity’s variation. Standard precalculus curriculum approach to a constant rate of change is often limited to solving linear equations and finding the slope of the “equation of the line.” It rarely focuses on the meanings of quantities, ratio, rate and constant rate of change. Therefore, it is important to discuss a conceptual analysis that discusses the productive ways of thinking to understand the idea of constant rate of change. Furthermore, it is relevant to design a hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004) that presents an order of learning goals for students with a series of tasks that promote the specific learning goals to understand the idea of a constant rate of change. The primary research question I want to address is- what understandings are foundational to understand the idea of constant rate of change?

Theoretical Background

Mathematics educators and researchers have investigated students' understanding of proportionality, rate, and ratio at various mathematical grade levels, with findings consistently revealing student difficulties in applying proportional reasoning when interpreting the rate of change in a modeling context (Tourniaire & Pulos, 1985; Doerr & O'Neil, 2011; Orton, 1983 & Yoon, Byerley & Thompson, 2015). Learning the idea of constant rate of change associates to the concept of rate, ratio, proportionality, changes in quantities, linear functions, and rate of change functions. According to Lamon (2007), early definitions of "ratio was considered a comparison between like quantities (e.g. pounds: pounds) and a rate a comparison of unlike quantities (e.g. distance: time)" (p.634), and over the years researchers have developed multiple definitions and distinctions of ratio and rate (Lesh, Post & Behr, 1988; Kaput & West, 1994). After working with students and teachers for a substantial amount of time, Thompson (1994), Thompson & Thompson (1994 & 1996) suggested that the distinction between rate and ratio depends on an individual's mental operations on how she comprehends the given rate and ratio context. Thompson (1994) defined ratio as a result of comparing two quantities multiplicatively. The two quantities that are compared multiplicatively, can be measured in the same unit or they can be measured in different units. In either case, if an individual conceives a mental operation of comparing two quantities multiplicatively, she conceives the result of the comparison as a ratio. Thompson (1994) defined rate as a reflectively abstracted constant ratio. When one conceives the idea of rate, she thinks about the rate as a characteristic of how quantities are covarying. A person reconceives a ratio as a rate when she applies the ratio to a different situation and think about that ratio as a rate that characterizes covariation between quantities. In another words, a ratio is a multiplicative comparison of the measures of two non-varying quantities and a rate is the proportional relationship between two varying quantities' measures (Thompson & Thompson, 1994).

Proportional reasoning as a theory interacts with the theory of quantitative reasoning (Thompson, 1988, 1990, 1993, 1994, 2011) as one conceives quantity as a measurable attribute of an object, then she conceives a multiplicative comparison of two fixed quantities. She conceives ratio as a result of the multiplicative comparison of the quantities. She conceives rate as a proportional relationship between the measure of two varying quantities. If the value of a quantity varies (or changes) in a situation, we say the quantity is a varying quantity, and if the value of a quantity does not vary in a situation, we say the quantity is fixed. When two quantities vary in relation to each other, the mental operations that support that dynamic images in students' thinking is referenced as covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). When a student engages in proportional reasoning, she simultaneously engages in covariational and quantitative reasoning. A student engages in proportional reasoning when she conceives the invariant relationship of quantities in dynamic situations or applies her understanding of proportionality in mathematical modeling situations. Constructing the idea of rate involves envisioning two quantities in a situation vary smoothly and continuously, and the changes (increases or decreases) in one quantity or variable's value is a simultaneous result of changes in another quantity or variable's value; and as the two quantities covary, the multiplicative comparisons of their measures remain proportional.

Conceptual Analysis

I used radical constructivism (Glaserfeld, 1995) as the theoretical perspective of this report. Radical constructivism claims that an individual constructs his own knowledge and it is not directly accessible to others. Glaserfeld (1995) also made a case that two individuals can reach a certain level of mutual agreement by providing compatible interpretations about each of their actions in the conversation. This idea of intersubjectivity seems to support the constructs conceptual analysis (Glaserfeld, 1995), hypothetical learning trajectory (Simon, 1995), that are relevant to investigate students' development of the foundational ideas in learning the idea of constant rate of change.

Thompson (2008) considered conceptual analysis as a tool to explain the useful ways of thinking for learning an idea. As one example of a conceptual analysis, Thompson (2008) described what is involved in understanding and learning the idea of constant rate of change and its role in understanding the idea of an average rate of change, linear functions, proportionality, and slope. In his conceptual analysis, he discussed the important role of quantitative, covariational, and proportional reasoning for learning the idea of a constant rate of change. Thus, if a student develops the deep meaning of constant rate of change, she should conceptualize the relationship between two quantities as continuous variation in their values, and the relationship remains proportional as two values of two quantities vary together.

I use the candle burning context (Carlson, Oehrtman, & Moore; 2016) to illustrate the productive ways of thinking of the idea of constant rate of change. This conceptual analysis describes what we have observed and hypothesize to be productive ways of thinking for learning the idea constant rate of change.

An 18-inch candle is lit and burns at a constant rate of 2.25 inches per hour. What is the length of the candle when it's been burning for 4.1 hours?

After reading the problem statement, it is necessary to realize that as the candle is burning since it is lit, the remaining length of the candle (in inches) is decreasing since the number of hours the candle is lit. The length of the candle (in inches) does not change if it is not lit. The remaining length of the candle (in inches) depends on the number of hours since the candle is lit. Therefore, the two quantities covarying together are the remaining length of the candle (in inches) and the number of hours since the candle is lit. Let *r* represents the remaining length of the candle (in inches) and let *t* represent the number of hours that have elapsed since the candle was lit. *r* can be written as an output of a function *f*, where $f(t)$ represents the remaining length of the candle since it was lit (in inches) in terms of the number of elapsed hours *t* since the candle was lit. We can say, $r = f(t)$ and where r is the output of the function f and f is a function of t. Here, r and t are variables that represent the varying values of the two covarying quantities- the remaining length of the candle since it was lit (in inches) and the number of hours since the candle was lit.

Representing the remaining length r of the candle since it was lit (in inches) in terms of the numbers of hours t since the candle was lit involves conceptualizing how quantities' values change together. The candle was originally 18 inches tall and burns at a constant rate of 2.25 inches per hour. To elaborate the problem statement, we can say for any change in the number of hours since the candle was lit (since $t = 0$) the change in the remaining length of the candle in inches is decreasing at a rate of 2.25 inches per hour. After *t* hours since the candle was lit the change in the remaining length would be *t* times as large as 2.25 inches less than 18 inches. There is a continuous variation in the number of hours since the candle is lit and a continuous variation in the remaining length of the candle since the candle was lit. The remaining length of the candle (in inches) is always the number of hours since the candle was lit times as large as 2.25 inches shorter than the original length of the candle. The expression $-2.25t + 18$ represents the remaining length r of the candle in inches and we can write a formula $r = -2.25t + 18$ in terms of t.

We can also write the formula using function notation, $f(t) = -2.25t + 18$, where t is the input of the function and independent variable that represents the number of hours since the candle was lit and $f(t)$ represents the functional relationship between r and t in terms of t, where r is the output of the function and dependent variable, and f is the function name. The function formula provides a description of how t and $f(t)$ vary together.

Moving forward, a student needs to conceptualize what is the length of the candle after it burns for 4.1 hours. Here, she needs to imagine the remaining length of the candle (in inches) when 4.1 hours have elapsed since the candle was lit. At this point, it is essential to conceptualize 4.1 hours of burning as a change of 4.1 hours since the candle was lit. In this context, the reference point is $(t_1, r_1) = (0{,}18)$ where r_1 represents the length of the candle before it was lit and t_1 represents the number of hours since the candle was lit. The reference point is a point in a context from which we imagine the variation in variable's value when the quantities are covarying together. As the change in the value of the remaining length of the candle since it is lit (in inches) remains constant with the change in the value of the number of hours since the candle was lit, we can represent the change in the value of the remaining length of the candle (in inches) since it is lit with Δr and the change in the value of number of hours since the candle was lit with Δt and $\Delta r = -2.25\Delta t$. The change in value of the remaining length of the candle since it is lit is decreasing with respect to the change in the value of the number of hours since the candle was lit. Therefore, the constant rate of change is -2.25 inches per hour. For any value of t, the change is away from 0 is also the value of t, $\Delta t = t - 0 = t$. Any value of t can be substituted into the function formula and is interpreted as a value of Δt changing away from $t = 0$. Similarly, the change in the value of the candle's remaining length (in inches) is a change in value of r, away from 18, $\Delta r = r - r_1 = r - 18$. We can write, $r - r_1 = -2.25(t - t_1) \rightarrow r - 18 = -2.25(t - 0) \rightarrow r = -2.25t + 18$ which is an equivalent form of the function formula $f(t) = -2.25t + 18$, where $r = f(t)$.

Therefore, if we want to know the length of the candle when it's been burning for 4.1 hours then $t = 4.1$ and $r = -2.25 * (4.1) + 18 = -9.225 + 18 = 8.775$, where the expression $-9.225 + 18$ represents the candle's length changes by -9.225 inches when the time elapsed by 4.1 hours and the candle is 8.775 inches long 4.1 hours after being lit.

The function $f(t) = -2.25t + 18$ represents a linear functional relationship between the remaining length of the candle since it is lit (in inches) and the number of hours since the candle was lit. As we vary t, the value of t is a change away from 0 and the change in length (in inches) is always -2.25 times as large as the change in elapsed time (in hours). The function represents a proportional relationship between the change in values of the remaining length of the candle since it is lit (in inches) with respect to the change in the number of hours elapsed since the candle was lit. The change in both quantities' values remain proportional for any amount of change in one quantities value with respect to another. When changes in one quantity's value are proportional to the corresponding changes in the other quantity's value there is a constant rate of change. Representing the relationship between both quantities values in a graphical context is always a straight line. Dynamic images of variation in the value of remaining length of the candle (in inches) since it is lit with respect to the variation in the value of the number of hours since it is lit is useful to track the proportionality in both quantities' values along with the linear representation of change in their values.

Hypothetical Learning Trajectory

I present a hierarchy of ideas and describe them as I conjecture these ideas to be foundational for understanding constant rate of change. Simon (1995) and Simon & Tzur (2004) discusses hypothetical learning trajectory (HLT) as an order of learning goals for students with a series of tasks that promote the specific learning goals to understand any idea. Based on Simon's idea of HLT we present a hierarchy of the ideas that are foundational to learning the idea of constant rate of change (Table 1). Table 1 elaborates the hypothetical order of introducing the ideas, the ideas or sub ideas that are foundational for understanding constant rate of change and the required student understanding for each of the section.

Hypothetical order of ideas	The foundational ideas for understanding constant rate of change	Desired student understanding
1.	Quantity: i. Varying or fixed quantity ii. Covariation iii. Changes in quantity's varying values	Given a problem statement, a student starts thinking about identifying fixed and varying quantities in the situation. She thinks about which quantities are varying independently or with respect to another quantity; or value of one quantity that does not change in the context. She also thinks about whether the varying quantities in the quantity changing together or not; and if two or more quantities are changing with respect to one another she will think about them as covarying quantities. Then she will think about how changes in one quantity's value is related to changes in another quantity's value.
2.	Representation: i. Variables ii. Expressions iii. Formula iv. Variables (delta notation) to express change in quantity's values v. Graphical representation	The student thinks about using variables to represent the value of varying quantities. She thinks about using variables, numbers or operators to express the value of a quantity as an expression. She conceives a formula as a tool for expressing the value of one quantity in terms of another, when the quantities are covarying. She thinks about change in a quantity's value as it is always away from a reference location. She represents the changes in quantity's values with delta notation. She thinks about presenting the relationship between quantities in a graphical representation. She will think about a cartesian coordinate system as a system where she can represent the values of the independent quantity in a horizontal axis, and the values of the dependent quantity in a vertical axis. She then thinks about a graph as a representation of how the covarying quantities are varying together.
3.	Times as large as comparison i. Ratio ii. Proportionality	She thinks about two numbers A and B, as A/B is a measure of A in units of B. She thinks if A/B as a ratio then A is A/B times as large as B (provided $B \neq 0$) and B is B/A times as large as A (provided $A \neq 0$).

	iii. Rate	She thinks two quantities are proportional if the measure of one quantity's value in units of the other's value is a constant as the two quantities vary together. Two quantities are proportional if the value of one quantity is always the same number of times as large as the value of the other quantity. She perceives rate as an instantaneous image of a constant ratio, when one of the two covarying quantity's value is measured in units of another quantity's value.
4.	Constant rate of change	She thinks about constant rate of change as relating two co-varying quantities, when changes in one quantity's value are proportional to the corresponding changes in the other quantity's value. She thinks about using variables x and y to represent the values of two quantities that change together, then if the quantities are related by a constant rate of change then $\Delta y = m. \Delta x$ where Δy represents the changes in y values and Δx represents the changes in x values. The changes in y's values are m times as large as changes in x's values.
5.	Linear graph	She perceives a graph representing constant rate of change is always linear with a slope of the constant rate. Any change in one quantity's value is proportional to the corresponding change in the other quantity's value when the quantities are changing at a constant rate. She thinks proportionality is a restriction on how the quantities are varying together.
6.	Function i. Function notation ii. Independent and dependent quantity iii. Functional representation	She thinks function f defines a rule of a function that expresses how a function's independent quantity and dependent quantities are related as their values change together. If there are two covarying quantities x and y that possess a functional relationship and f is a function, then $y = f(x)$ and $f(x)$ is represented in terms of x. Here, she thinks x represents the independent quantity and y is the dependent quantity. f is a function name, values for x are input values and values for y for corresponding x values are output values of the function f.
7.	Constant rate of change function or linear function	She thinks a function that represents relationship between changes in the values of two covarying quantities and changes in one quantities value remain constant with corresponding changes to another quantities value, the function is a constant rate of change function or linear function.

Table 1. The hypothetical order of ideas that are foundational for understanding a constant rate of change

Implications and Limitations

One of the implications of the proposed HLT (Simon & Tzur, 2004) is to develop lessons and homework problems in a way where a student will build her foundational understandings of the idea of constant rate of change by reflecting upon her actions and the effects of those actions when working on the problems. As a methodological implication the HLT is a sophisticated tool to design sequenced tasks to conduct teaching experiments (Steffe & Thompson, 2000). The teaching experiment will focus on characterizing students' ways of thinking as they complete tasks designed to support their understanding of the foundational ideas in learning the idea of constant rate of change. The episodes of teaching experiments will inform the researchers to revise the HLT and make adjustments on the tasks to support new hypotheses as the results will unfold. This paper is limited to the theory of conceptual analysis and hypothetical learning trajectory in discussing the foundational ideas for understanding constant rate of change. I have designed a teaching experiment to investigate students' thinking of the foundational ideas for understanding the constant rate of change. The results of the teaching experiment will be addressed in the future. The conceptual analysis and the HLT will be useful to other researchers and teachers to design sequenced lessons or tasks to promote learning the idea of constant rate of change. The following figure 1. presents an example of an instructional sequence of tasks that informs the foundational ideas for understanding constant rate of change-

Sarah started a trip with 15 gallons of gas in her car's tank. Her car uses gasoline at the constant rate of 0.05 gallons per mile. Sarah also drove at a constant speed of 60 miles per hour until the car's tank was empty.

Q. What quantities are varying in this situation?
Q. How these quantities change together?

Q. What does it mean to say that Sarah's car uses gasoline at the constant rate of 0.05 gallons per mile?
Q. If Sarah's car uses gasoline at the constant rate of 0.05 gallons per mile, then how far Sarah can drive with one gallon of gasoline?
Q. How much gasoline (in gallons) is consumed when Sarah drove a distance of 5 miles?
Q. How much gasoline (in gallons) is consumed when Sarah drove a distance of 0.08 miles?
Q. Write a formula that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the distance s (in miles) Sarah has driven since she has started the trip.

Q. What does it mean to say Sarah drove at a constant speed of 60 miles per hour until the car's tank was empty?
Q. Write a formula that represents Sarah's distance s (in miles) in terms of the amount of time t (in hours) since she has started the trip.
Q. How much gasoline (in gallons) was consumed when Sarah drove for one hour maintaining a constant speed of 60 mph?
Q. How much gasoline (in gallons) was consumed when Sarah drove for 1/3 of an hour maintaining a constant speed of 60 mph?
Q. Write a formula that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the amount of time t (in hours) since she has started the trip.

Q. How does the amount of gasoline g (in gallons) remaining in the car's tank varies with respect to the amount of time t (in hours) since she has started the trip?
Q. How far (in miles) from Sarah's starting point she would drive until the car's tank is empty?
Q. How long (in hours) Sarah would drive until the car's tank is empty?

Q. Construct graphs – (a) that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the distance s (in miles) Sarah has driven since she has started the trip.
(b) that represents Sarah's distance s (in miles) in terms of the amount of time t (in hours) since she has started the trip.
(c) that represents the amount of gasoline g (in gallons) remaining in the car's tank in terms of the amount of time t (in hours) since she has started the trip.

Figure1. A sequence of tasks for learning the foundational ideas of constant rate of change.

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