Influences on Problem Solving Practices of Emerging Mathematicians

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Mathematical problem solving research that focuses on the development of problem solving practices can provide mathematics educators the tools with which to foster the use of such practices in their students. This study hopes to identify influences on the development of emerging mathematicians' problem solving technique. Emerging mathematicians from an undergraduate real analysis course and first-semester graduate course in linear algebra were invited to a sequence of two interviews, during which they completed problem solving tasks and reflected on their growth as mathematical problem solvers. In particular, they were asked to expound on formative experiences that affected the development of their problem solving strategies. Participants reported ways that both teaching mathematics and learning mathematics from skilled instructors influenced the way they solved novel problems.

Keywords: Problem solving, Teaching, Graduate students

The body of research on mathematical problem solving is multifaceted. Studies in this field, for example, might hope to explain the nature of problem solving, generate a framework for analyzing problem solving, or catalogue the problem solving behaviors of expert mathematicians (cf. Schoenfeld, 1989; DeFranco, 1996; Selden et. Al., 1999; Carlson & Bloom, 2005). Resourceful mathematics educators could leverage any one of these studies, regardless of its stated purpose, in service of second, implicit goal: to better understand how to foster problem solving growth in their classrooms.

One might suspect that, to help mathematical novices grow as problem solvers, one might only have to teach them to replicate the problem solving strategies employed by experts. But as noted by Silver (1985),

> This is specious reasoning and such advice may be counterproductive to learning. Surely the behavior of experts is highly efficient and often elegant, but the polished, mature problem solving behaviors of experts fail to reveal the successive approximations that preceded these mature procedures (pg. 251).

The study described in this report aims to capture these successive approximations (and thereby better understand the expert problem solving methods in which they culminate) by asking emerging mathematicians to identify which experiences contributed to the ongoing development of their problem solving strategies. Here, we define an *emerging mathematician* as a student of mathematics who understands foundational content knowledge but has had limited opportunities for refinement of their mathematical practice. Over the course of a semester, participants self-reported experiences both as teachers and as students of mathematics that influenced their process for solving novel problems.

Background and Theoretical Perspective

One hallmark of classrooms that successfully encourage problem solving is the breadth and connectedness of the mathematics knowledge they impart (Arcavi et al., 1998; Collins et al., 1988); another is that this knowledge is also robust and thoroughly justified (Arcavi et al., 1998; Schoenfeld, 2014). While these qualifications might be lumped together in order to assert that expert problem solvers simply have more content knowledge, this is an oversimplification.

Certainly, experts have more content knowledge (DeFranco, 1996). However, content knowledge cannot be leveraged for problem solving if it is *inert* (Whitehead, 1929; as cited in Renkl et al., 1996), that is, confined only to applications that are similar to the specific instructional circumstances under which it was introduced. The difficulty for some mathematics students (primarily, those with a preponderance of inert knowledge) to apply familiar concepts to novel situations is sometimes referred to as the *transfer problem*; researchers have developed many pedagogical strategies that hope to address this issue. Those that fall under the *situated learning* model of instruction (e.g. Bransford et al., 2012; Spiro et al., 1991; Campione et al., 1988, Collins et al., 1988) hope to overcome the transfer problem by introducing mathematics concepts already embedded in a novel, real-world context from the outset.

Problems of transfer, however, are not only relegated to students who cannot solve real-world problems with abstract mathematics. Research has also shown that even otherwise successful calculus students have significant difficulty applying their textbook knowledge of the subject to completely abstract (but entirely novel) problem solving situations (Selden et al., 1999; Selden et al., 1994). In response, some pedagogical models describe instructional strategies that teach both content knowledge and powerful thinking strategies, allowing the application of knowledge to solve unfamiliar problems (e.g. Liljedahl, 2016; Schoenfeld, 2014; Fiorella & Mayer, 2015). Illustrations of such classrooms are multifaceted and necessarily describe not only of the type of content delivered but also the method of delivery, type and frequency of corresponding assessment, layout of the classroom, and appropriate questioning strategies. One belief embodied by these models is that students are more likely to develop successful patterns of thought and effective problem solving strategies when they are actively engaged both with the content and with each other (see also Silver, 1985; CBMS, 2016). A notable example of such a classroom is Schoenfeld's problem solving course, taught at UC Berkeley and analyzed, in part, in Arcavi et al. (1998). Through an authentically collaborative mathematical environment, this course emphasized effective modes of mathematical thought that included instruction on both metacognitive strategies and the application of specific heuristics.

Methodology

Setting and Participants

The study took place at a large, urban, public, and Hispanic-serving university. In order to recruit emerging mathematicians, participants were solicited from the population of upper-division undergraduate mathematics majors and first-semester mathematics graduate students. Undergraduate participants came from an undergraduate analysis course that has an "introduction to proofs" course prerequisite; that is, these undergraduates had experience with foundational techniques for mathematical justification but had not had extensive experiences refining them through prolonged or advanced application. Graduate participants came from a first-semester graduate course in linear algebra. Their status as first-semester graduate students acknowledges a baseline experience and competency with a broad spectrum of mathematical technique; however, as beginning graduate students, they had not yet encountered the demands of mathematics graduate courses, mathematical research, and exploration of unsolved problems.

In the first week of the Fall 2020 semester, both groups of participants responded to a short questionnaire (administered at the beginning of a class period corresponding to one of the respective courses described above) in which they were asked to rate certain problem solving techniques using a Likert scale (from "1 – Strongly Disagree" to "6 – Strongly Agree"). For an example of such questionnaire items, see Figure 1 below.

	When solving a difficult math problem, it is helpful to...						
1.	Read the problem carefully	1	2	3	4	5	6
2.	Try and imagine what a solution might look like	1	2	3	4	5	6
3.	Consider multiple different ways in which the problem might be solved	1	2	3	4	5	6

Figure 1. Select Likert scale items from the preliminary questionnaire.

Participants also indicated their interest in participating in a sequence of two interviews. The first interview was scheduled as soon as possible so that it would take place close to the beginning of the semester; the second interview occurred at the end of the same semester. For interview protocol and procedure, see the following section. A total of 11 undergraduate students and 20 graduate students responded to the initial questionnaire, and of these, 4 undergraduate students and 8 graduate students participated in the interview sequence. Due to the small number of respondents, the interviewed participants necessarily represented a convenience sample. Information on interview participants is provided in Table 1, below.

Table 1: Interview participant information

Pseudonym	Gender	Age Bracket	Population
Will	Male	18-22	Undergraduate
Riley	Female	18-22	Undergraduate
Brady	Male	18-22	Undergraduate
Verna	Female	33+	Undergraduate
Anne	Female	18-22	Graduate
Greg	Male	23-28	Graduate
Taylor	Other	28-33	Graduate
Mia	Female	18-22	Graduate
Sara	Female	18-22	Graduate
Frank	Male	33+	Graduate
Julie	Female	18-22	Graduate
Carol	Female	23-28	Graduate

Data Collection and Analysis

Each interview was recorded and lasted between 45 and 90 minutes. During an interview, participants completed a sequence of three tasks intended to prompt problem solving engagement. They were also asked to comment on their decisions and thought processes while working on the task sequence. For an example of an interview problem, see Figure 2 below.

Problem 2. *What value(s) of α maximize the number of solutions to the following system of equations?*

$$x^2 - y^2 = 0$$
$$(x-\alpha)^2 + y^2 = 1$$

Figure 2. The second task given during the first interview, adapted from Schoenfeld (1998, pg. 98).

At the end of the first interview, participants also elaborated on their responses to the initial questionnaire. This process was facilitated by the interviewer, who asked them to expand on initial questionnaire responses that were sufficiently different from the average response for their population. During the second interview, after completing the requisite sequence of three tasks, participants also completed the same questionnaire to which they had already responded at the beginning of the semester. This time, the interviewer prompted them to respond to questionnaire items for which their second response was significantly different than their first, i.e., if the response was two or more points different or when it indicated any shift from agreement to disagreement (or vice versa).

The goal of the questionnaire analysis conducted during interviews was to provide participants with a chance to reflect on the development of their opinion towards particular problem solving strategies; to supplement this goal, a number of interview protocol questions were asked at the end of each interviewer. For example, these questions included: (1) How has your approach to problem solving changed (over time/ in the past semester)? (2) Has any singular (class/ person/ event) significantly affected your approach to problem solving (in the past semester)? How? (3) Describe the particular qualities of your favorite math course that made it stand out.

Responses to interview questions and any discussion of questionnaire items were initially transcribed verbatim. The select quotes provided in the section below have been lightly edited for clarity by removing speech disfluencies, except when doing so would alter the tone or meaning of a sentence. The transcripts were then analyzed using open and axial coding (Strauss & Corbin, 1998) that aligned with techniques from thematic analysis (e.g., Braun & Clarke, 2006; Nowell, Norris, White, & Moules, 2017). An initial coding pass was conducted by both researchers, who then met to compare their work; codes were expanded, subsumed, or redefined appropriately until both researchers were in agreement.

Results

Below, we introduce two prominent themes that emerged during interviews: participants recognized when development of their problem solving strategies had been instigated by their experiences both while teaching and while learning from influential teachers.

Growth while Teaching

Across interviews, participants identified that teaching experiences served to augment their originally surface-level understanding of mathematics. For example, Mia explained how teaching had increased her proclivity for sense-making:

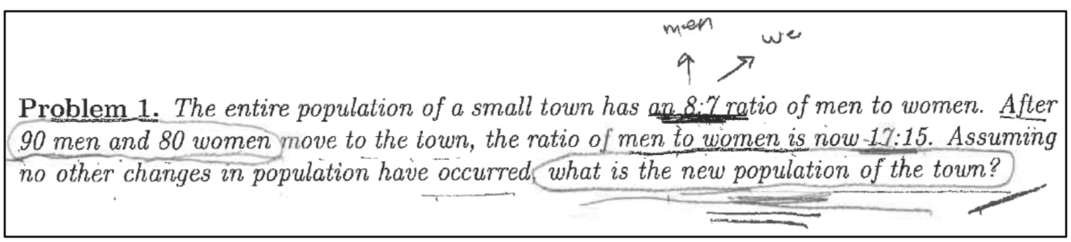
> Before, when I used to do math but I didn't understand something, I feel like I would just skip it. And then move on, take it as: okay, this is fact. […] Teaching my calculus course, since I have to go *so* in-depth with the students because they ask *so* many questions, I feel like—now, I feel like I get more caught up on the things that I don't understand as well. […] Now when I understand things, I *understand* them.

Brady identified this phenomenon as having to “over-learn” a topic that he planned to teach, a practice which made him aware of an important pedagogical dichotomy tied to sense-making: “You can memorize a formula, or you can actually understand what the formula's doing.” Anne echoed Brady’s sentiment, explaining that adopting the former strategy was “the reason why people need tutoring, I feel: they don't understand why you're doing what you're doing. And so, I can't just be like, well, use this formula. I don't know why it works, but it does. It just confuses them more. So it's made me have to figure out why I was doing things in previous courses as well.” Several participants found that preparing for teaching not only increased the depth of their mathematical understanding but also its breadth; especially when dealing with students of varying mathematical competencies, participants recognized that, as teachers, they needed to consider multiple ways to approach one problem or explain one concept. Carol observed that “While going for the teaching, I see sometimes, like: what are the other options to solve this? What can be the different methods?” and that this practice of planning ahead and weighing multiple options carried over into her own problem solving.

Additionally, some participants found that teaching led them to discover specific techniques or explanations that could be applied to their own areas of study. Anne noted that teaching “things to help them [her students] figure out what to do on a problem, when they see it, helps me to do that.” She was not the only participant who adapted her own problem solving strategies to reflect what she recommended to her students: Frank noted that he was “more attuned to” making sure his work “flows in a logical way” after interacting with students whose work was disjointed and not well-supported with mathematical justification. Verna taught her students the importance of carefully reading and interpreting the text of a given problem:

> I want to help them to understand the question. So I—like, for me, I read it but go with them over the question and ask them: look what they ask here. They figure out, oh, this is what they ask? Like this. So I make them to understand the question and what they ask about. And they have the knowledge, but they didn't know. That's why—that’s helped me.

Appropriately, Verna appeared to be especially conscientious of the need to read interview tasks carefully. Figure 2 below illustrates a sequence of annotations indicative of her approach to interview tasks; this included identifying important numerical data, isolating the question being posed to better anticipate the form of her solution, and clarifying any unfamiliar mathematical notation.


men
we

Problem 1. *The entire population of a small town has an 8:7 ratio of men to women. After 90 men and 80 women move to the town, the ratio of men to women is now 17:15. Assuming no other changes in population have occurred, what is the new population of the town?*

Figure 3. An example of Verna’s propensity for carefully attending to the text of a given task. Problem 1 preceded Problem 2 (see Figure 1) in the first interview.

Riley described tutoring a student to whom she would sometimes give the solution to a problem. Then, she would prompt the student to “justify why I got that, exactly. What am I doing, and why does it relate to this problem?” She mentioned, in her second interview, that thinking about her own work from a similar perspective (especially when that work was proof-based) was sometimes helpful when approaching a difficult problem. In both interviews, Riley lived up to this claim. For example, she attributed her difficulty solving the task pictured in Figure 1 to the fact that she was “not completely certain what the answer’s supposed to look like,” and so could not apply this strategy. On the other hand, in a different task that required her to inscribe a square in an arbitrary triangle, Riley successfully applied an approximation of her working backwards strategy by starting with a square and instead attempting to verify whether she could circumscribe an arbitrary triangle about it.

Growth while Learning

Many participants also commented on those of their own instructors whose pedagogical decisions they perceived as influential in their problem solving development. Some instructors contributed directly to this development by giving the participant new strategies or heuristics for tackling novel problems. For example, Riley described the lasting impression of a favored instructor's teaching strategy:

> It's one thing for a professor to give you a function and give you a problem, give you an example, and show you how it works. You could mimic that, a lot of the time. But to have a professor that gives you a problem that you haven't seen before, and gives you certain tools that will help you solve the problem, and so you know that those tools are there, you'll be able to solve the problem a lot easier than someone that's mimicking. They won't take that away.

Other participants were more specific about the problem solving heuristics that they had learned. Verna learned the value of providing thorough mathematical justification. With respect to one influential professor, she said, "When she give us a problem, she told us to go step by step and don't, like—don't ignore *this*. This [has] some meaning. *This* [has] some meaning. You must understand what you [are] talking about." Brady, in reference to the same instructor, remembered the same lesson: "But with most of our proofs, it felt like it was pretty much line by line, you know? […] And I felt like, in the beginning, that that was very important. Because now, I don't just skip a few lines of work without really thinking about it first." This behavior was in evidence during the first interview, wherein Brady solved a task that required him to cut an arbitrary right triangle into two isosceles triangles. There, he drew an arbitrary rectangle and attempted to appeal to the mathematical properties of its diagonals. Brady was able to indicate places during this justification process where he relied on visual intuition to substantiate a claim; he explained that a truly rigorous explanation would not be so informal.

Riley explained how her previously tentative approach to novel problems was affected by one instructor who encouraged her to, "even if you're not sure about going about a problem, draw something. Put *something* down. Because it's probably worth it in the end. Because you're thinking of it." Carol had also been encouraged to draw diagrams by several instructors who valued visual representations, and she noted that this was particularly helpful for reframing familiar proofs in more interesting, intuitive ways; as an example, she recalled one professor who used diagrams to illustrate the relationship between Cauchy and convergent sequences. During her work on the task pictured in Figure 1, Carol drew ten discrete visualizations as tools for sense-making and justification. This was more than the rest of the participants combined; a selection of Carol's visualizations are provided in Figure 4 below.

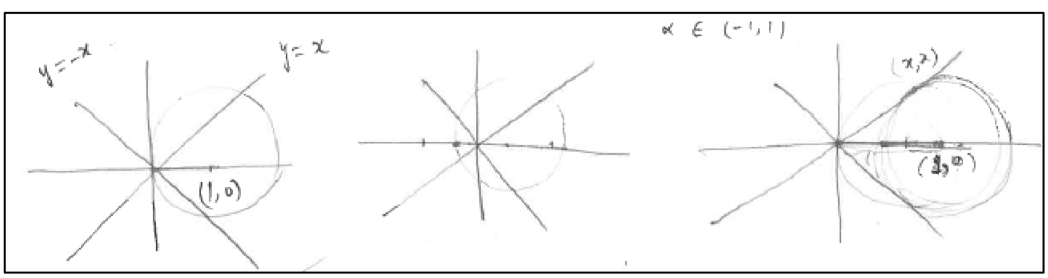


Figure 4. Some of Carol's diagrams, depicting three different cases corresponding to different values of α.

Carol also noted that her professor's visualizations were more engaging for students: "It was not just writing on the board and we have to copy that." Other participants also found previous instructors memorable for their ability to engage students by giving them agency in the creation of mathematics. Brady remembered an instructor who called on the class directly to answer important mathematical questions, including having "us do proofs up on the board and stuff. The students of the class. And that kind of, I guess, added a community feeling to the class." The epistemological responsibility of students to generate their own explanations even extended to one of Frank's professor's office hours, and Mia recalled one professor who "challenged you to think by yourself. […] Like, he didn't *tell* you the answer. He made you think it through." Sara explained the value of classrooms in which students themselves were involved in the creation of mathematics when she said that students who are "part of the conversation" are better equipped to explain the phenomena that they have learned. To substantiate this claim, she drew upon her experience with a recent professor who encouraged students to generate their own definitions for important concepts; from a problem solving perspective, this challenged Sara's preconception that mathematics was a rigidly defined subject wherein problems had prescribed methods of solution.

Conclusion and Implications

From participant interviews, a number of themes arose with respect to problem solving development. Graduates and undergraduates alike thought that teaching mathematics helped them to recognize and leverage more effective problem solving techniques. In particular, participants saw increased value in drawing diagrams, justifying their methodology, and carefully making sense of the problem text. Most graduate students also felt that their teaching experiences gave them a deeper understanding of the material, both because of the pedagogical requirement that they know several effective explanations and because of the opportunity to reacquaint themselves with older content. As observed in Selden et al. (1999), incremental growth in facility with content over a period of repeated exposure is not unusual; students in constant contact with calculus, for example, can still take years to be able to apply that content in creative ways. Furthermore, repeated exposure from a teacher's perspective might foster a deeper understanding of mathematical content. Such deeper understanding is a fundamental aspect of many pedagogical frameworks aimed at improving problem solving (e.g. Schoenfeld, 2014). In participants' work, these conceptual advantages allowed them to consider multiple approaches to a given problem and apply a diverse array of heuristics; furthermore, they felt comfortable justifying why their chosen methodology was both appropriate for the circumstances and mathematically sound.

On the other hand, undergraduates were more likely than graduate participants to recall specific problem solving strategies taught to them by instructors. These strategies were identical, though, to two of the previously noted strategies learned through teaching: visualizing to make sense of difficult concepts and justifying every claim. As students, both graduate and undergraduate participants also recognized that their most influential instructors tended to give them the opportunity to discover and take ownership of the mathematics content. This agency engaged students, mirroring recommendations from literature (Schoenfeld, 2014; Liljedahl, 2016) and giving participants better tools with which to justify their mathematical reasoning. It also instigated an affective shift; participants who constructed their own mathematical meanings felt more confident in constructing their own approaches to novel problems. That is, participants reported feeling less compelled to remember and apply a "correct" method after experiencing the flexibility of mathematics first-hand in an influential classroom environment. Such instruction may thus promote the ability of students to transfer content knowledge from the original introductory environment to novel problem solving situations.

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