The Circle Schema: An Example of Schema Interaction in College Geometry

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There is a need to address student understanding of the role of definitions in undergraduate mathematics, and research is needed to determine pedagogical strategies that facilitate development of this understanding. In geometry, research shows students can better develop their understanding of concepts and their definitions by observing properties and making conjectures in non-Euclidean geometry. In this study, students learned concepts in Taxicab geometry in a College Geometry course in which theory in Euclidean geometry is the primary focus. The triad of stages of schema development and a model of schema interaction were used as frameworks in the analysis of student responses to a questionnaire and follow-up interview about the definition of circle in Taxicab and Euclidean geometry. As a representative illustration of a level of schema interaction, insight into the thinking of one student is presented in this report. Pedagogical suggestions are provided as a result of this data analysis.

Keywords: Definitions, Geometry, Taxicab, Circle, Schema

Introduction

Studies show that many preservice and even in-service teachers have insufficient understanding of the high school mathematics curriculum (Speer et al., 2015) and, more specifically, mathematical definitions (Chesler, 2012; Moore-Russo, 2008; Leikin & Winicki-Landman, 2001; Zazkis & Liekin, 2008). Edwards and Ward (2008) claim this is true for many mathematics majors and that this should be addressed in undergraduate mathematics, as research is needed to determine pedagogical strategies that help facilitate student's understanding of the concept of definition. Research shows by exploring concepts and definitions in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). One example of a non-Euclidean geometry in which students can explore concepts is Taxicab geometry, which is the geometry that is the result of measuring distance as defined by the L_1 norm. Siegel, Borasi, and Fonzi (1998) and Dreiling (2012) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it easier for students to reason and thus abstract concepts. For this report, we present results and discussion on the following research questions pertaining to student understanding of **Circle**: (a) How do students adapt their understanding of relationships among concepts associated with the definition of *circle* from Euclidean geometry to Taxicab geometry and vice versa? (b) How can using a model of schema interaction for the *circle in Euclidean geometry schema* and *circle in Taxicab geometry schema* help to describe the overall, underlying structure of the *circle schema* in students' minds?

Theoretical Framework

In APOS Theory, a constructivist framework based on Piaget's *reflective abstraction*, there are four different stages of cognitive development: Action, Process, Object, and Schema (Arnon et al., 2014; Dubinsky, 2002). A description of these levels of cognitive development in a general sense are omitted for the sake of space and detailed descriptions of the different levels of cognitive development associated with particular concepts in Euclidean and Taxicab geometry (like **Distance, Circle,** and **Midset**) are provided in Kemp (2018) and Kemp and Vidakovic

(2018, 2019, 2021). Arnon et al. (2014) stated that to describe certain learning situations, considering the schema structure may be necessary. As a result of the progression of APOS-based research, analyzing this structure may help to explain "why students have difficulty with different aspects of a topic, and may even have different difficulties with the same situation in different encounters," (p. 110). Considering the development of a student's schema has proven to lead to a deep understanding of how he or she reasons when confronted with a mathematical problem situation - how a student uses certain evoked components of a schema and relates them to one another when presented with these situations can reveal the structure of this schema and its development. Arnon et al. (2014) state that there is a need to investigate the development of schemas and how they are applied in mathematics.

To investigate the development of the *circle schema*, we utilize "the triad" of stages of schema development proposed by Piaget and García (1989). In studies such as Clark et al. (1997), Cotrill (1999), McDonald et al. (2000), Baker et al. (2000), and Trigueros (2000, 2001), researchers found the addition of the triad to their analysis helped to paint a better picture of how the components of schemas work together in certain circumstances. In particular, the reader's attention is focused to Baker et al. (2000) and how the authors define an overall calculus graphing schema in terms of the interaction of two schemas, as it is the first model of schema interaction described in detail. The authors describe the relationship between what are named the *interval schema* and the *property schema* and analyze common student errors when solving "an atypical calculus graphing problem," (p. 558). It is also noted Trigueros (2004) provides a second model of schema interaction for the solutions of systems of differential equations.

Below, descriptions are provided of the stages associated with the triad of development of the *circle in Euclidean geometry schema (cEg)* (which is the *Euclidean geometry schema* as it is evoked within the *circle schema*) and the mental constructions believed to be necessary to achieve these stages. Similarly, the evoked *Taxicab geometry schema* within the *circle schema* will be referred to as the *circle in Taxicab geometry schema (cTg)*. A genetic decomposition is defined as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). For this study, a genetic decomposition was developed to identify the different ways the *cEg* and *cTg schemata* interact with one another in a student's mind. The *circle schema* is described as its components may be evoked within the interaction of the *Euclidean geometry schema* and *Taxicab geometry schema,* using the framework and genetic decomposition presented in Baker et al. (2000) as a model. This genetic decomposition was used in data analysis and will be referred to as the *cEg-cTg schema interaction*. The genetic decompositions presented in Kemp (2018) and Kemp and Vidakovic (2018, 2019, 2021) were also used in analysis to help inform the researchers of the underlying structure of students' *circle schema* and how this schema is rearranged, or *accommodated,* to form new relationships among its components. For the representative example presented in this report, a brief description of the level of schema interaction will be provided. This example is chosen as an illustration of the interaction of schemata since the student appeared to accommodate her *circle schema* during the interview. This student provided insight as to how connections can be made to facilitate the development of the *circle schema* and ways to deepen student understanding of the role of mathematical definitions.

The three stages of schema development are prefixed Intra-, Inter-, and Trans- and each of these stages is described in general in Arnon et al. (2014). For length, we only provide descriptions of these stages in relation to the *cEg schema*. The stages of schema development are similar for the *cTg schema*. A student operating at the **Intra-cEg** exhibits this by viewing the components of the circle schema as isolated structures. A circle in Euclidean geometry is analyzed in terms of its properties either geometrically or algebraically (e.g. - it is round). Explanations of properties are local and particular. At least an action conception of the concepts of **Euclidean Distance, Radius, Center**, and **Locus of points** are necessary mental constructions to operate at this stage. A student operating at the **Inter-cEg** can exhibit this by forming relationships among the isolated ideas from the Intra-cEg stage and making connections between geometric and algebraic properties of a circle in Euclidean geometry. At least a process conception of some of the concepts of **Euclidean Distance, Radius, Center,** and **Locus of points** (and evidence that he or she can coordinate at least two of them) are necessary mental constructions to operate at this stage. A student exhibiting that she is operating at the **Trans-cEg** has constructed an awareness of the completeness of the circle in Euclidean geometry schema and can "perceive new global properties that were inaccessible at the other levels," (Baker et al., 2000, p. 559). The student coherently understands and can describe the construction of a circle and the structure of the equation for a circle in Euclidean geometry and how these are a result of the definition of a circle. It is necessary for a student to have at least a process conception of all **Euclidean Distance, Radius, Center,** and **Locus of points** concepts and has coordinated all combinations of these processes (which coordinates all geometric/algebraic representations).

Due to length, we provide here only a partial genetic decomposition for the *cEg-cTg schema interaction.* In general, this model includes nine levels of schema interaction that result as the two schemata of *circle in Euclidean geometry (cEg)* and *circle in Taxicab geometry (cTg)* interact with one another at various stages. These nine stages are shown in Table 1 below and each of these is described in detail in the full genetic decomposition. As an example, in the genetic decomposition, the mental structures necessary for a student to operate at an Inter-*cEg* stage of schema development and Intra-*cTg* stage of schema development, and what the interaction of these schemata look like at these different stages (which we refer to as the *Inter-intra* level of schema interaction) are described.

Table 1. The nine levels of cEg-cTg schema interaction.

<table>
<tr><td></td><td></td><td colspan="3">circle in Taxicab geometry schema (cTg)</td></tr>
<tr><td></td><td></td><td>Intra-</td><td>Inter-</td><td>Trans-</td></tr>
<tr><td rowspan="3">circle in Euclidean geometry schema (cEg)</td><td>Intra-</td><td>Intra-intra</td><td>Intra-inter</td><td>Intra-trans</td></tr>
<tr><td>Inter-</td><td>Inter-intra</td><td>Inter-inter</td><td>Inter-trans</td></tr>
<tr><td>Trans-</td><td>Trans-intra</td><td>Trans-inter</td><td>Trans-trans</td></tr>
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Methodology

This research study was conducted at a large university in a College Geometry course during a Fall semester, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were both undergraduate and graduate students enrolled in the course. The textbook used in the course was College Geometry Using the Geometer's Sketchpad (Reynolds & Fenton, 2011), written based on APOS Theory and the ACE Teaching Cycle (Asiala et al., 1996). The material of the course covered concepts and theorems in Euclidean geometry often seen in a College Geometry course and included Taxicab geometry for four class sessions at the end of the semester. Written work from the semester and videos from the in-class group work and discussion during the Taxicab geometry sessions were collected as data. After the semester but before final exams, semi-structured interviews were conducted with 15 of the 18 students enrolled in the course who voluntarily signed up to participate in the interviews. The following questions from the questionnaire were relevant to this data analysis, and are a subset of the questions asked before and during the interview:

1. Define and draw an image (or images) that represents each of the following terms however you see fit: Circle, Distance.
2. For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$
   (i) Euclidean distance is given by $d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   (ii) Taxi distance is given by $d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|$
   Using the grids below, illustrate each of these two distances. Be as detailed as possible in labeling them.
3. Is the following definition true in both geometries? Explain. "The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
4. Using the grids below, sketch the following circle in both geometries: Circle with center at $C(3,3)$ and radius $r = 2$.

In general, there is no unique ideal response for most of these questions since the main goal of these interviews was to get students to elaborate on their thought processes as they responded to these questions. However, Figure 1 below shows how a student could illustrate what is asked in Question 2 and Question 4. For Question 3, this definition holds in both geometries, but the circles will visually appear different as a result of the way in which distance is defined.

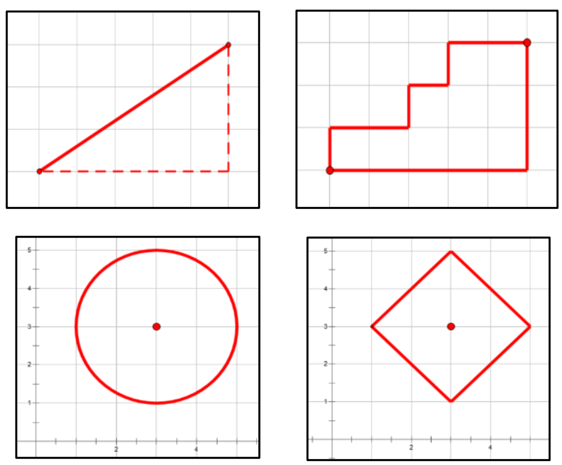

Figure 1. Visual representations of Euclidean and Taxicab distance (top row) and a Euclidean and Taxicab circle each with center (3,3) and radius 2 (bottom row).

By asking questions about both Euclidean and Taxicab geometry on the questionnaire as well as probing from the interviewer, students were presented with an atypical task, just as the participants in Baker et al. (2000). For analysis of data from these interviews, using the idea of schema interaction allowed for a deeper analysis of each student's make up, or structure, of the *circle schema*. Considering each of the concepts within this schema (**Distance, Radius, Center,** and **Locus of points**) helped to describe these structures. There was also a need to consider each of these concepts within both Euclidean geometry and Taxicab geometry, how each student understood these concepts within these geometries both algebraically and geometrically, and how this resulted in their overall understanding of the definition of **Circle.** From this information and using the genetic decomposition for the *cEg-cTg schema interaction*, all 15 students' work and responses were analyzed. Due to space limitations, only one case of one level of schema interaction is presented in this report, and more information about schema interaction can be found in Kemp (2018). What follows is a description and example of a student exhibiting the *trans-trans* level of schema interaction between the *cEg* and *cTg schemata* associated with the *circle schema.* This analysis will focus on what components of one student's *circle schema* were evoked to justify the responses she provided on a questionnaire prior to a follow up interview and her exploration of these concepts during the interview.

Results

A student is exhibiting operating at the trans-*cEg*, trans-*cTg* (trans-trans) level of schema interaction when the student has generalized the concept of **Circle** completely and has a coherent understanding of how the construction of a circle and the equation of a circle is a direct result of the definition of a circle. That is, the student has generalized the structure of the equation for a circle geometrically and algebraically in both Euclidean and Taxicab geometry. In terms of APOS Theory, the student has successfully constructed new processes geometrically and algebraically by coordinating all of her **Distance, Radius, Center,** and **Locus of points** processes. There is a complete coordination of the student's *cEg* and *cTg schemata*, and the student would be able to evoke any necessary components of these schemata to coherently talk about a circle in both geometries. There were three students out of the 15 who participated in interviews who exhibited evidence of operating at the trans-trans level of schema interaction. As a representative illustration for this level, we provide the APOS Theory and schema-interaction based analysis of Parker's responses to the questionnaire and prompts during her interview. We present a description and representative example of this level of schema interaction as this student provided evidence that she was able to accommodate her *circle schema* during the interview to begin operating at a trans-trans level of schema interaction. Through this accommodation, she provided unique insight into her thinking.

Parker was both a graduate student enrolled in the MAT program and a secondary mathematics teacher and was interviewed with two other graduate students. Parker's definition of a circle written on the questionnaire was the "set of all points equidistant from a given point known as the center." On the part of the questionnaire that asked if the definition of a circle provided held in both Euclidean and Taxicab geometry, Parker responded "Yes, the given distance just looks different in reference from the center." This provided evidence that while evoking her geometric representation of **Circle** and saying, "distance just looks different," Parker was aware that the choice of metric is the distinguishing factor between the visual appearance of a circle in Euclidean and a circle in Taxicab geometry. Parker's illustrations of a Euclidean and Taxicab circle from the questionnaire are shown in Figure 2, below. Although she was not asked to explain how she constructed her geometric representations of the circles, many details can be deduced from her illustrations.

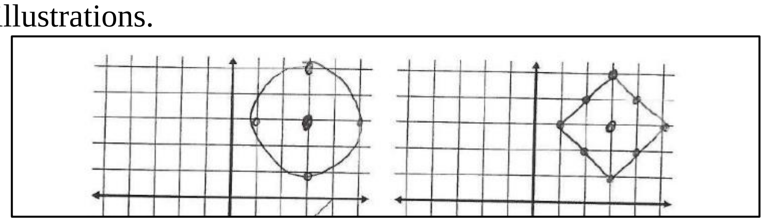

Figure 2. Parker's illustrations of a Euclidean and Taxicab circle on the questionnaire.

In her sketch of the circle in Euclidean geometry, she plotted the four points that were two units away from the center on the vertical and horizontal from the center then most likely "connected" these points based on how she knew the circle would look. For her sketch of the circle in Taxicab geometry, in addition to plotting the same four points on the vertical and horizontal from the center, she also plotted four more points that were two units away from the center and used these eight points to construct her circle. This is an indication that she likely used her understanding of the definition of a circle and the Taxicab metric to construct this circle, and that she had formed a coherent understanding of the geometric representation of a circle. In each interview, the students were prompted to try to write the equation of the circles they had drawn in Euclidean and Taxicab geometry. During conversation between the interviewer and the other two students in this group interview, Parker was writing down the equation for Taxicab distance under her illustration of a circle in Taxicab geometry, seen in Figure 3. It is noted that anything in red ink indicates it was written during the interview (and black writing was written while filling out the questionnaire prior to the interview). She interrupted the conversation between the interviewer and the other students and said the following.

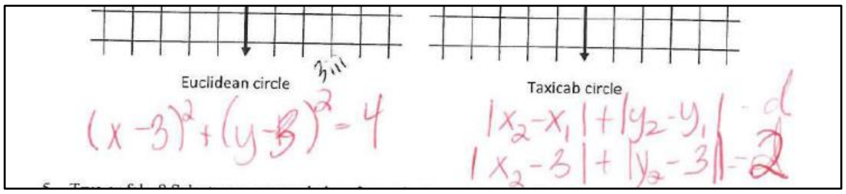

Figure 3. Parker's equations for the Euclidean and Taxicab circles she drew, written during the interview.

Parker: This might be wrong… I wrote the formula for…calculating distance for Taxicab, and then I know…this has to be compared to the center which is (3,3) so… if my distance is 2, doesn't that make it?...Is equal to 2. So let me change that…'Cause then like this point up here is [(3,5)] and if we plug in 3 and 5…3 minus 3 is 0, 5 minus 3 is two, so that would mean our distance is 2, which means it's on the circle.

Note that she used the variable d, and then substituted the value of 2 (the radius) for d, which is what indicated she understood how distance and the measure of the radius was involved in the equation of a circle. Parker implied here verbally that she knew the distance formula should be related to the radius when she says, "this has to be compared to the center". Specifically, she indicated she knew if the distance from the center to all of the points on this circle is the same, then this distance should be equal to the radius of the circle. She then used an arbitrary point to verify that a point she knew *geometrically* was on the circle, would also satisfy her *algebraic* representation of the circle, saying, "…our distance [between the center and (3,5)] is 2, which means it's on the circle". In this moment, we believe Parker had accommodated her *circle schema* and made connections between the geometric and algebraic representation of a circle in Taxicab geometry. It appeared Parker had generalized her understanding of the definition of a circle and was aware how the representations of a circle geometrically and algebraically are a result of the definition of circle. Parker provided evidence here that she was coordinating her **Distance, Radius, Center,** and **Locus of Points** processes across her *cEg* and *cTg schemata* to articulate her reasoning for writing the equation of her Taxicab circle in this way.

The interviewer then asked Parker if she could she draw a circle using a different metric if she were given one (and to describe how she would approach that task). Parker responded confidently, "yeah… figure out what the metric is asking…and what it's… well we eventually get our radius, and then we can…," and she began moving her finger in the air like she was starting at a center point and drawing outward. In further elaboration, she said "…the unit circle…and the radius needs to be one, so we have to figure out from the metric…what would yield an answer of one and if we do that then we can sketch the circle." Without explicitly stating she would construct multiple radii in order to sketch the circle, she implied this by her motions in the air with her finger, since it is interpreted that she was constructing multiple radii using some arbitrary metric. Further, she described that for a unit circle, she would use the new metric to find the points that would be one unit away from the center ("would yield an answer of one"). During this exchange, Parker provided evidence she had successfully constructed new processes geometrically and algebraically by coordinating all of her **Distance, Radius, Center,** and **Locus of points** processes. She appeared to have a complete coordination of her *cEg* and *cTg schemata* and was able to coherently talk about a circle, not only Euclidean and Taxicab geometries, but in an arbitrary space. For these reasons, Parker provided evidence that by the end of the interview she had formed a complete, coherent understanding of the underlying structure of the *circle schema* and was operating at the trans-trans level of schema interaction.

Discussion and Concluding Remarks

Parker exhibited evidence that she had generalized the definition a circle geometrically and algebraically in both Euclidean and Taxicab geometry. She also indicated she would be able to draw a unit circle using an arbitrary metric and, based on her reasons for writing the equation of the circle in Taxicab geometry, we believe she would be able to also write the equation for that circle. All three students who exhibited evidence of operating at the trans-trans level of schema interaction were able to explain the structure of the equations of a circle in both geometries in general terms and how it related to their illustrations and the definition of a circle. This is the main difference between the students operating at the inter-inter level and the trans-trans level. Some of the students operating at the inter-inter level of schema interaction could recall the Euclidean circle equation and could write the equation of a circle in Taxicab geometry, but mainly relied on "copying" the format of distance formulas. In contrast, the students operating at the trans-trans level showed they could clearly discuss how the distance formula is involved with this equation as a result of the definition of a circle. By making connections between the algebraic and geometric representations of concepts in both Euclidean and Taxicab geometry, students were able to generalize their understanding of this definition. There appeared to be a positive relationship between the amount and depth of these connections and the extent to which students were able to generalize the definition of a circle. In the example provided here, by visualizing and algebraically defining a circle in an atypical context, such as Taxicab geometry, Parker continued to construct and re-construct relationships between components within her *circle schema*. This is consistent with Dreiling (2012), Hollebrands, Conner, & Smith, (2010), and Jenkins (1968) in regard to the development of her understanding. This is evidence that more activities meant to help students form these connections between geometric and algebraic representations in geometry should be used in classroom instruction.

There is a lack of research that uses of the stages of the triad in APOS Theory (for examples, see Clark et al. (1997) and McDonald et al. (2000)), but even fewer studies that use the idea of the "double triad" or schema interaction. A model of schema interaction was used in this study and one level of schema interaction was described in detail in this report, using the original model by Baker et al. (2000) as a guide. Baker et al. (2000) determined that analyzing their data using APOS Theory without the triad or schema development hindered the researchers from understanding the thought process of many of the participants. Similarly, by involving a genetic decomposition for a model of schema interaction into the analysis of this study, the researchers were able to better describe the structure of the participants' *circle schema* and their overall understanding of the definitions of various concepts in geometry. It is believed that the genetic decomposition of this schema interaction developed for this study could be used as a model for how students can assimilate any metric into their *circle schema*, although further research would need to be done to validate this.

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