Are We Sharing the Same Interpretations For “Prove” And “Show?”: Comparison of An Instructor’s and Students’ Interpretations

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Proof is a medium of mathematical communication and has distinct characteristics related to it. Among various characteristics, we focus on the words in proving tasks prompts (PTP), such as “prove,” “show,” “justify,” or “explain” because students’ arguments can be changed depending on the keyword, which can impact assessing their learning when the students and the instructor do not share the same interpretation for the same keyword in PTP. Thus, in this study, we observed an Introduction to Proofs course to see which and how the keywords in PTP are presented to students and interviewed the instructor and a focus group interview of her three students. The finding indicates that the instructor’s interpretation and students’ interpretations were different. Although students are adjusting their pre-existed meaning for those words to the context, this study suggests that instructors cannot assume that students share the same meaning of the keywords in PTP with them.

Keywords: proving, proof, argumentation

Learning a discipline is deeply related to the discourse used therein so that one can fully participate (Lemke, 1990; Lew & Mejía-Ramos, 2019; Schleppegrell, 2007)**.** Specifically, to communicate using mathematical arguments, it is important for students to learn the discourse used and to understand the expectations of them in such situations. However, upon examining typical types of problem statements used in Introduction to Proof (ITP) courses, we find a variety of words used to prompt proving tasks such as “prove [a claim],” “show [a claim],” “give an explanation,” or “justify your answer.” Thus, we posed the following questions: “Are these words prompting the same task? Are they interchangeable?” Students are asked to create or evaluate mathematical arguments in response to varying prompts such as “prove/show/justify/explain a claim.” What often occurs is that instructors enter the discursive moment with their own interpretations of the prompts for proving tasks while students enter with their own (possibly differing) interpretations. Hence, we spotlight the keywords in proving task prompts (PTP), such as *prove*, *show*, *justify*, or *explain*, as an important area for getting students to think about what they are asked to do.

Of course, one might ask why the issue of such a small aspect of proving tasks is of concern in undergraduate mathematics education. Depending on the prompt, students’ interpretations affect the logical structure of the arguments and how they present their work. Moreover, if one’s goal is to assess how students understand proving, assessing their understanding of proving becomes difficult if students think that the words for prompting proving are interchangeable. For example, depending on how students interpret *show*, some might think that they can provide some examples or can use graphs to make their point, whereas others may view *show* as a synonym to *prove*. As these keywords in PTP can change the structure of the arguments, we think it is important to focus on how people interpret these words. Especially, we focus on whether a teacher and their students would interpret the words similarly. Therefore, in this study, we investigated how an instructor and her three students interpret the keywords, *prove*, *show*, *justify*, and *explain*, in PTP and whether they interpret similarly or differently.

Background and Theoretical Framework

Several researchers have raised questions about the meanings of keywords in PTP, such as "show," "prove," "justify," "convince," or "explain," used to prompt proving tasks (e.g., Dreyfus, 1999; Hersh, 1993; Hwang & Karunakaran, 2019). For instance, Alcock (2013) posited that mathematicians consider "prove" and "show" to be synonyms when they prompt proving tasks. Dreyfus (1999), however, questioned the word "show" when it is used to prompt argumentation as it asks for students to provide a formal proof or an inductive argument. This question is supported by a study of Calculus I students' interpretation of the words "prove" and "show" used to prompt proving tasks (Hwang & Karunakaran, 2019, 2020**.** Hwang and Karunakaran (2019, 2020) suggested that students might think the word "prove" necessitates a narrower set of appropriate and formal options than the word "show". Their finding suggested that students think the prompting word "show" could prompt an inductive argument even if they think that argument is not appropriate for "prove." Moreover, they found that responses to the prompting word "show" need to focus on demonstration and can include visuals or the measuring of figures.

As another example, Mejía-Ramos and Inglis (2011) studied student engagement with the words "proof" and "prove" used to prompt proving. They used semantic contamination as their framework, and the research was based on their assumption that meanings from daily use of the words might be able to influence students' engagement with tasks. Their finding suggested that students responded more casually to the verb "prove" than the noun "proof," which aligns with their research on the usage of the two words in English. The finding also suggested people's usage of words in daily life affects their responses to mathematical questions.

Although very few researchers have examined the meaning of keywords in PTP, the need for further research about the prompting words could be found from some of the existing studies. Researchers questioned and/or reflected on how they worded their prompts for interviews or survey questions. For instance, Liu and Manouchehri (2017) reflected that changing wordings of questions might help to get a better result regarding the gap of student understanding concerning roles of proofs; thus, they called for further research on how these wordings affect the research. Also, sometimes researchers did not explicitly reflect on wordings to prompt students' responses during the interview tasks, but their results indicated that students might interpret those words differently. Bieda and Lepak (2014) purposefully avoided using the prompt "prove" and used "convincing" as a word to study students understanding, considering their population, middle school students. Their research results implied that students distinguish "show" and "tell (explanation)" when they are asked to choose "convincing" arguments. As another example, a student in Healy and Hoyles (2000) indicated that "examples could be a "proof," although not an "explanation" (p. 418), which again implied that students distinguish between proofs and other types of justifying arguments. This result possibly means that students would have differently responded to prompts that were worded differently. Hence, again, we claim that understanding linguistic aspects of the prompts in proving tasks are important because it is one of the starting points in solving the problem and building their arguments (Mejía-Ramos & Inglis, 2011).

In this study, we use the framework by Stein et al. (1996) as our theoretical framework that guides our study design and data collection. According to the framework by Stein et al., (1996) a task evolves throughout three different stages: when a teacher or task designer develops the task; when the teacher sets up the tasks to the classroom; and when the task is enacted in the classroom. They claim that the sequence of this evolution eventually influences students' learning. We consider that teacher's interpretation as a kind of teacher subject matter knowledge that can influence the task set-up as well as viewing students' interpretation as a factor influencing the task implementation. Therefore, according to the framework, if a teacher and their students do not share the same interpretation of the keywords in PTP, then it can lead students to possible misunderstanding of mathematical proving.

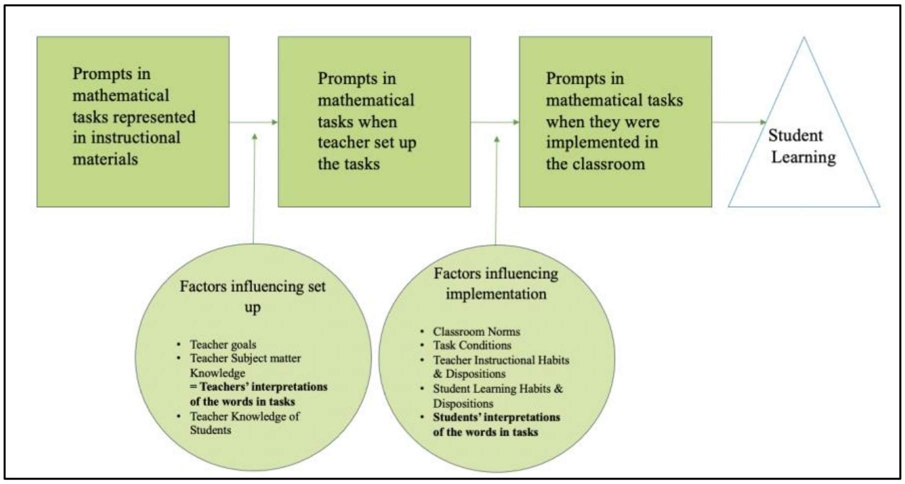

Figure 1. The modified mathematical framework by Stein et al. (1996)

Methods

The setting of the reported study is an ITP course in a large public midwestern university. We chose the ITP course because instructors and students in the course need to engage with proving tasks frequently. Therefore, an instructor of the course would have some interpretations of the keywords in PTP as well as at least some understanding or expectation of how students would react to the keywords. For that reason, Alycia (all names are pseudonyms), who was one of the six instructors teaching the course in spring 2020, participated in the study. Alycia was a mathematics doctoral student teaching this course for the first time as the instructor of record, but she had served as a teaching assistant for this course during the previous semester. The data collection with her consisted of two cycles of a half-hour-long pre-interview, two classroom observations, and an hour-long post-interview. During the pre-interviews, the first author asked her about her teaching plan for the week such as proofs that she would present to students or the problems that students would do. Alycia's voice was recorded during the classroom observation and the first author took notes on what she wrote on the board. Then, during the post-interviews, there was a brief discussion of what she did and whether there was a relevant case related to keyword in PTP as well as two interview tasks–one for each post-interview.

After the data collection with Alycia was finished, three students (Ryan, Leo, and Tyler) from Alycia's class participated in a focus group interview. The reason we chose the focus group interview was for participants to have a similar situation to their classroom because the ITP course in the university used groupwork as one of its major practices. During the focus group interview, they solved one proving task, discussed some questions related to PTP, and then collectively tackled the task used in Alycia's second post-interview. In this second task, participants were asked to sort six already generated arguments for the same statement "$x^2 + 4x + 18 > 0$ for all real numbers x." to one or more of the keywords in PTP, *prove, show, justify, explain*, or *none of the above*. The six arguments had different characteristics such as using a graphing calculator (A3), using empirical examples (A4), using algebraic expressions

(A1), or using different mathematical contents (A2, A5, and A6) (some examples in Figure 2). All interviews were video-recorded, and the observations were audio-recorded.

Once the data collection was finished, all of the interviews and relevant parts of observations were transcribed. In the first round of coding, sentences were categorized depending on the focus keywords (*prove, show, explain,* and *justify*). At this stage, for Alycia's data, we distinguished between her responses that were about how she would herself interpret the prompts and those that were about her expectations for her students' responses. Under each keyword, descriptive codes were developed. After the second round of coding, narratives of each keyword in PTP for an instructor and the focus group were created. To compare an instructor's interpretations and students' interpretations, we categorized relevant codes with three characteristics of mathematical arguments from Stylianides (2007): *set of accepted statements, modes of argumentation,* and *modes of argument representation*. Set of accepted statements refers to mathematical statements accepted by the mathematical community where the author meant to present the argument; modes of argumentation refer to the logical structure of an argument such as using inductive reasoning or deductive reasoning; and the modes of argument representation indicates the form of argument such as linguistic, graphic, or symbolic aspects. We posit that the keywords in PTP can influence on each of these three characteristics, especially the latter two.

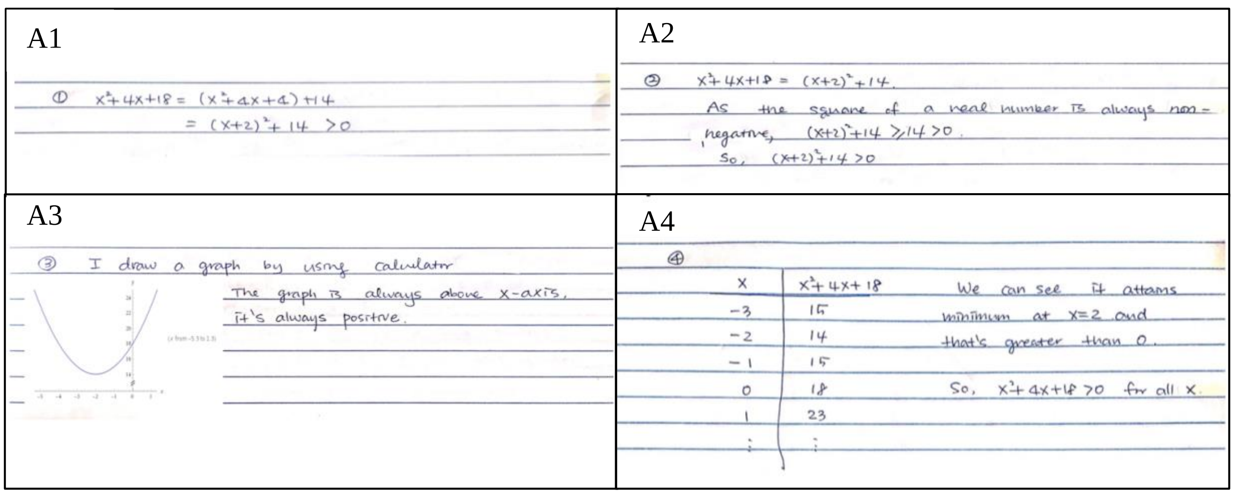

Figure 2. Some Examples from The Task in Which All the Participants Engaged

Findings

In this paper, because of the space constraints, we focus on *prove* and *show* which are the most common keywords in PTP used in proof-based mathematics courses. From our data analysis, we found that Alycia refers to *prove* and *show* as synonyms, but her students do not consistently agree. First, we present the data from the instructor, Alycia, and then we present the students' data, which we compare to Alycia's responses.

Alycia's Interpretations

Alycia herself interprets *prove* and *show* as synonyms as commonly used in the professional mathematical community and wanted her students to consider them that way as well:

> Alycia: if someone said, "*show* me that this is true," I feel like I still need to go through every scenario. And part of that is just probably just because of my training [as a mathematician].

This quote shows that due to mathematical experience, she became to think that *prove* and *show* are synonyms, because both of the words ask her to consider all possible scenarios. In the first post-interview, she claimed that she had tried to imply the message to students by switching up the words. But in the later interview, she said she will state the synonymity explicitly to students. And in the observation, we found that she vocalized the following expectation to her students:

> Alycia: So, *show* or *prove*. So, in your everyday life, these might mean different things, other professors that you encounter in your life, they may mean different things, *show* or *prove*. But if I ask you to *show* something, I'm asking you to *prove* it. [If] I ask you to *prove* something, I'm asking you to *show* it.

As well as wanting her students to know the synonymity of *prove* and *show*, Alycia also expected them not to use example-only based argument (empirical arguments) or calculators. In addition, she wanted her students to have good proof writing skills where she viewed a proof as a response to *prove* and *show*.

Alycia's Anticipation of Her Students' Responses

Although she emphasized and wanted her students to know the synonymity of the two words, she anticipated that students might answer differently to *prove* and *show*. Alycia thought that responses to *prove* would be more formal than responses to *show*. As part of it, she thought that students might use various forms, such as tables or graphs, when answering to *show*. In addition, she also wondered if students would feel less intimidated when they saw *show* or more stressed because they are not sure what *show* meant compared to when they saw *prove*:

> Alycia: …when people think of a proof or something like that in the mathematical sense, I think that just, I think, I'm sure plenty of people's blood pressure spikes when they hear that [*prove*] …. is if I asked them to *show* something, will that give them anxiety because they don't know what I'm asking for? You know? So, so it is kind of like this double-edged sort of for like, sure. Proof might sound more intimidating … So then if I, so, so if I ask them to show something but maybe they don't know what that means, well then give them more anxiety, then if I asked them to prove something and that just sounds more intimidating.

Hence, although Alycia viewed and expected students to know that *prove* and *show* are synonyms, she expected some students to struggle in interpreting and responding to different PTP words.

Students' Interpretations

From the focus group interview with her students, it seemed her anticipation of how students would respond to those words has some relationship with how a subset of her students interpret the words. Students were aware that *prove* and *show* are similar and that similarity being consistent in the class. However, they needed to remind themselves that those are asking the same things:

> Leo: So, if it says, just like, "Show why this isn't true," then you can ... Your first reaction, a lot of times, is going to be just like, "Okay, I'm just going to write down the counterexample and move on." But we really, then you have to like, step back and like, "Wait a second. They're not asking for that. They're asking for a proof.

Descriptions similar to this one appeared repetitively. Student participants were reminding themselves that the class they are in is "Transitions," where their instructor told them *prove* and *show* are synonyms. The following is the description by Leo when the interviewer asked them what the word *show* means to them:

> Leo: Sometimes it's, show is, in some cases, *show* can mean like, give an example of something, and then, in some cases, like in [course number of Transitions], a lot of times it ["show"]'s going to mean *prove*. So, you have to just make the judgment, yourself, of what they're really trying to ask.

Leo again referred to the course and mentioned that they are asked to produce the same arguments to *show* as if they are responding to *prove* in the context. Ryan and Tyler agreed with this description. However, as Leo indicated, they viewed *show* to mean something else in different contexts, which is related to how Alycia anticipated students would respond to *show*.

Also, related to Alycia's wondering related to affective aspects of the two words *prove* and *show,* students claimed that they felt different for both of them. During the interview, I brought up the case when Alycia mentioned in the class that *prove* and *show* are synonyms, and the following were their reactions:

Interviewer: … I remember Alycia, one time, said, "Show and prove are the synonyms in the class."

Leo: Yeah, but like-

…

Leo: Yeah, you hear that, and, but it's like, you see it on the paper, you don't think of them the same way.

Ryan: Yeah.

Interviewer: Mm-hmm (affirmative).

Tyler: If somebody says show, or prove, you're like, "Well, all right. I'm going to be doing pretty much the same thing," but there's a different feel, I think, to both of them.

Ryan: Mm-hmm (affirmative).

Although the three students did not say and I did not ask further at this point how they felt differently, during the interview they interpreted that *show* is more casual or provided "a little bit of breath of relief" than *prove* did. However, in general, it seems that they were adjusting their reaction to the word *show* as what the mathematical community that they belong to interprets it as.

Comparison of Alycia's and The Three Students' Interpretations

Although students are being enculturated to the professional mathematical community and both populations' interpretations looked similar, the responses to the task indicated some differences. When looking at Alycia's interpretation of the four keywords, deductive argumentation was the base requirement for any of them because she chose none of the above for non-deductive argument (A3 and A4; see Table 1). However, students included A3 as a possible answer for the *show* PTP, which means that deductive argument was not necessarily the requirement, but they focused on the graphical mode of argument representation. Similar to this, student participants generally paid lots of attention to the mode of argument representation such as focusing on defining variables, explaining every step, or making a conclusion, when they were deciding if something can be a response to the *prove* in PTP. This finding is related to why students' choice and Alycia's choices were different for the choice "prove," and seems to be related to Alycia's standpoint that she wanted to teach them to have a good habit of writing proofs.

Also, when comparing the choices for *show*, two of them are common A5 and A6, but the others do not overlap with each group's responses–A5 and A6 were deductive arguments using different mathematical knowledge. Considering A1 is an argument with only algebraic work and A3 is with graphs, it seems students still think concepts that *show* can have a variety of forms. Interestingly, the choices did not overlap with what Alycia anticipated her students might choose. Therefore, although (or as) students are learning how the PTP words are used in professional mathematical communities, their interpretations of the words were different than the instructor's interpretations and expectations.

Table 1. *Choices of the prompts*

	Prove	Show	Justify	Explain	None of the above
Alycia for herself	A2, A5, A6	A2, A5, A6	A1	A2	A3, A4
Alycia for students' possible responses	A2, A5, A6	A2, A4 (general students), A5, A6	A1, A3 (only first line), A4	A3, A4, A5 (nervous students)	
Students	A6	A1, A3, A5, A6	A2, A5, A6	A2, A3, A5, A6	A4

Discussion

Our study indicates that some students' interpretations of keywords in PTP were different than how their instructor interpreted the words. During the interview without the tasks, it seemed their meanings were similar, but their choices were different when we asked them to decide a proper PTP. More specifically, students distinguished *prove* and *show*, and thought that *show* can have a variety of meanings, unlike *prove*. This result can relate to Mejía-Ramos and Inglis's study (2011) when they found that student responded differently to the subtle difference between "*prove"* and "*give a proof*." As students have seen these keywords in PTP in their lives other than mathematics and even in other mathematics, the meanings can be transferred to the new mathematical contexts that they face.

However, in our study, we could observe that the students are adjusting their pre-existing meaning before the ITP course to what their instructor interprets it as, based on what the student participants mentioned as possible meanings in the other contexts such as calculus courses. But, given that this data collection is a month after the course started, students need some time to adjust their meaning of the keywords in PTP. Our study indicates, therefore, that interpreting the keywords in PTP as an instructor wants them to do might not be tedious for students requiring some extra time or work. This is because students need to learn if and so how three characteristics of mathematical argument (Stylianides, 2007) change depending on the PTPs. In addition, we also found that students' choices and their instructor's anticipated choices of the PTPs were different, which means that although the instructor is aware of the possible difference, it is not easier for them to anticipate how students would think about the responses or how students might respond to a given PTPs. That is, instructors also need some time and work to understand how their students are viewing the keywords in PTP and communicate with them about the possible issue.

These results suggest that the keywords in PTP can lead to students' responses being different than what task designers want, which can influence the assessment of student work; hence, task designers should be careful about the word choice in PTP especially when they do not have chance to share their meanings with task-doers such as in the case of national standardized tests or certain research settings. The data set in this study is not large, but generalization of the result was not the goal of the study, although the result that students see the words *prove* and *show* as different was found from our previous study as well (Hwang & Karunakaran, 2020). In the future, how these keywords in PTP influence students' responses in test settings on a large scale or how these keywords have been used in research settings might help the mathematics education community to see the importance of PTP.

References

Alcock, L. (2013). *How to study as a mathematics major*. Oxford University Press.

Bieda, K. N., & Lepak, J. (2014). Are you convinced? Middle-grade students' evaluations of mathematical arguments. *School Science and Mathematics*, *114*(4), 166–177. https://doi.org/10.1111/ssm.12066

Dreyfus, T. (1999). Why Johnny can't prove (with apologies to Morris Kline). *Educational Studies in Mathematics*, *38*, 85–109. https://doi.org/10.1023/A:1003660018579

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428. https://doi.org/10.2307/749651

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*(4), 389–399. https://doi.org/10.1007/BF01273372

Hwang, J., & Karunakaran, S. S. (2020). Students' Interpretations of the prompts for proving tasks: "prove" and "show." In S. S. Karunakaran, Z. Reed, & A. Higgins (Eds.), *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 260–267).

Hwang, J., & Karunakaran, S. S. (2019). Students' responses to differing prompts for reasoning and proof tasks. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1123–1124).

Lemke, J. L. (1990). *Talking science: language, learning, and values*. Ablex Publishing Carparation.

Lew, K., & Mejía-Ramos, J. P. (2019). Linguistic conventions of mathematical proof writing at the undergraduate level: Mathematicians' and students' perspectives. *Journal for Research in Mathematics Education*, *50*(2), 121–155. https://doi.org/10.5951/jresematheduc.50.2.0121

Liu, Y., & Manouchehri, A. (2017). Middle school children's mathematical reasoning and proving schemes. *Investigations in Mathematics Learning*, *6*(1), 18–40. https://doi.org/10.1080/24727466.2013.11790328

Mejía-Ramos, J. P., & Inglis, M. (2011). Semantic contamination and mathematical proof: Can a non-proof prove? *Journal of Mathematical Behavior*, *30*(1), 19–29. https://doi.org/10.1016/j.jmathb.2010.11.005

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, *23*(2), 139–259. https://doi.org/10.1080/10573560601158461 To

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: an analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488. https://doi.org/10.3102/00028312033002455

Stylianides, A. J. (2007). Proof and proving in mathematics education. *Journal for Research in Mathematics Education Journal for Research in Mathematics Education*, *38*(3), 289–321. https://doi.org/10.1007/978-94-007-2129-6