Making Connections: Advanced Perspectives on School Algebra

Keith Gallagher
Simon Fraser University

Anna Marie Bergman
Simon Fraser University

Fostering conceptual understanding is a main goal of mathematics education. Yet, students do not often see connections between topics. The recognition of connections can be fostered by making them explicit in instruction and by encouraging students to struggle with the relevant mathematics. Instances in which knowledge from one domain is brought to bear in another domain are examples of transfer; in particular, backward transfer describes the ways in which learning new information influences prior knowledge. We solicited narratives from individuals with a strong mathematics background describing instances during which they recognized a connection between two mathematics topics. We present two of these narratives which contain instances of backward transfer connecting advanced mathematics to the binomial theorem. These narratives suggest that backward transfer is also facilitated by explicit instruction and productive struggle. We discuss implications for education of backward transfer that is induced by learning advanced mathematics, particularly for pre-service teachers.

Keywords: backward transfer, conceptual understanding, binomial theorem, connections

Introduction

Broadly construed, mathematics teaching focuses on two central aspects: conceptual understanding and procedural knowledge. Procedural knowledge fixates on students' abilities to accurately apply mechanical techniques, including algorithms and formulas as well as algebraic manipulations and routine proof techniques. Conceptual understanding, however, links procedures with one another and with deeper meanings. Our research focuses on conceptual learning: specifically, we explored the influence of learning advanced mathematics on conceptual understanding of school algebra.

Recent research has explored the formation of cognitive connections between advanced mathematics and school mathematics (e.g., Murray et al., 2017; Wasserman, 2018; Weber et al., 2020). In particular, questions have been raised about the benefits of taking advanced mathematics courses such as combinatorics and linear algebra for pre-service teachers who will likely never teach the content of these courses. In this paper, we share personal narratives from two individuals who recognized connections between advanced mathematics and the binomial theorem, a common topic in high school algebra courses. We then discuss factors that may have precipitated the formation of these connections and conclude with a discussion of the potential impacts of advanced mathematics study on the understanding of fundamental mathematics concepts.

Theoretical Framing and Research Question

Conceptual Understanding

We adopt Hiebert and Grouws' (2007) definition of *conceptual understanding* as "mental connections among mathematical facts, procedures, and ideas" (p. 380). This is distinguished from *skill efficiency*, or "the accurate, smooth, and rapid execution of mathematical procedures" (ibid., p. 380). Combined, conceptual understanding and skill efficiency form the backbone of modern mathematics instruction, yet Brophy (1997) notes that we have much more knowledge of how learners develop skills than we do of how they build concepts. Although it has been shown that different kinds of teaching differentially support the learning of these two aspects of mathematics, there is not a simple correspondence between a single method of teaching and a single outcome; for example, there is not a causal relationship between expository teaching and rote (procedural) learning, nor between discovery teaching and meaningful (conceptual) learning (Ausubel, 1963; Hiebert and Grouws, 2007).

Nevertheless, from their meta-analysis of empirical studies on measures of student learning, Hiebert and Grouws (2007) identified two key features of teaching that promote conceptual understanding. First, teachers and students should attend *explicitly* to concepts by "treating mathematical connections in an explicit and public way" (p. 383). Examples of these kinds of activities include "discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas" (p. 383). Second, students should *struggle* with important mathematics. Hiebert and Grouws use the term *struggle* to mean that "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent... The struggle we have in mind comes from solving problems that are within reach and grappling with key mathematical ideas that are comprehendible but not yet well formed" (p. 387).

Backward Transfer

Connections among content may be established by means of *transfer*, the process by which "learning in one context or with one set of materials impacts on performance in another context or with other related materials" (Perkins & Salomon, 1992, p. 3). In experimentation, transfer is a rare phenomenon, especially *far transfer*, which occurs between contexts that are only distantly related in the mind of the learner (Barnett & Ceci, 2002; Perkins & Salomon, 1992; Sala & Gobet, 2017). Hohensee (2011) and Hohensee et al. (2019) extended the notion of transfer to include *backward transfer*, defined by Hohensee (2011) as "the influence on prior knowledge by the acquisition and subsequent generalization of new knowledge" (p. 13). For example, Hohensee et al. (2019) studied the influence of participating in an instructional unit on quadratic functions on high school algebra students' views of linear functions.

Although transfer is considered rare in experimental settings, two mechanisms for its occurrence have been identified (Perkins & Salomon, 1992). *Reflexive*, or *low road*, transfer, occurs when a well-practiced routine is triggered by familiar stimulus conditions; *mindful*, or *high road*, transfer, occurs as a consequence of a deliberate search for connections.

Research Question

In light of the research on conceptual understanding and transfer, we investigated the following research question: How does the learning of advanced mathematics influence conceptual understanding of school mathematics? In the coming sections, we discuss the personal narratives of two individuals who made conceptual connections between school algebra and more advanced areas of mathematics. These connections led to deep mathematical insights and new perspectives on elementary mathematics, which we conceptualize as instances of backward transfer. We discuss these connections in terms of Hiebert and Grouws' (2007) key features for promoting conceptual understanding as well as Perkins and Salomon's (1992) mechanisms for transfer.

Making Connections

We collected narratives from a total of 14 individuals, all of whom hold at least a master's degree in mathematics or statistics. We asked these individuals to tell us a story about an experience in which they have "suddenly recognized a connection between two mathematical concepts [they] hadn't noticed before, particularly an experience that helped [them] understand some aspect of mathematics better or in a different way." Some individuals told of connections formed between different parts of the standard K-12 curriculum (e.g., the Pythagorean theorem and the Euclidean distance formula) or between different topics from advanced mathematics (e.g., Taylor polynomials and the comparison test for series convergence). We took particular interest in those narratives that told of connections between advanced mathematics and the basic mathematics taught in school. In this section, we share two stories of learners making connections between advanced mathematics and basic algebra. Because these stories are unique to those who experienced them, we have endeavored to present the following narratives as they were presented to us so as to preserve the storytellers' voices. We withhold our analyses of these narratives until the subsequent section, Theoretical Connections in the Literature.

Robin – Connections to Combinatorics

At the time when she shared her story, Robin was a graduate student working toward a PhD in mathematics education, having already completed a master's degree in mathematics. Her story is about a connection she made between the binomial theorem, which she learned in high school, and combinatorics, which she learned later at university.

The binomial theorem relates combinations $\binom{n}{k}$, often read as "n-choose-k," with the expansion of binomials via exponentiation: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, where $n \geq 0$. Robin learned the binomial formula in her high school algebra course, but she described feeling unsure about what role choice played in this equation. She agreed that these combinations produced the correct coefficients when she worked out examples, and she could even see that these combinations yielded the entries in Pascal's triangle, but when she thought about $\binom{n}{k}$, she wondered: "What are we choosing here? Why does something have to be 'chosen'?" And was it a coincidence that the coefficients in the binomial expansion just happened to correspond to the values of $\binom{n}{k}$?

When Robin was later enrolled in a course in combinatorics at university, she connected the "choice" with the "jumps" she made while expanding a binomial expression. She explained this by writing $(x+y)^5$ as $(x+y)(x+y)(x+y)(x+y)(x+y)$, and drawing arcs connecting the x in the first binomial with x's and y's from the later binomials, as in Figure 1. It became clear that the combination was, in fact, counting the number of ways one could "choose" exactly k x's from the n possible ones. She noted the "nice symmetry" that this creates in the resulting polynomial.

$$(x+y)(x+y)(x+y)(x+y)(x+y)$$

Figure 1: "Jumps" indicating one way of choosing three x's when expanding a polynomial, as described by Robin.

This imagery also spurred Robin to notice why it must be true that $\binom{n}{k} = \binom{n}{n-k}$ – that one can choose k x's or *not* choose *n-k* x's in the same number of ways. Given five copies of $(x +$

y), one can choose exactly 3 x's in the same number of ways that one can *not* choose exactly 5-3=2 x's. Furthermore, Robin already knew the fact that the coefficients generated using the binomial theorem corresponded with the entries in Pascal's triangle, but this realization helped her to see the connection between the rule for generating the terms of Pascal's triangle and the combinatorial statement of Pascal's formula: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.

Martin – Connections to Linear Algebra

Martin holds a PhD in mathematics and is a university professor of mathematics and statistics. Martin, like Robin, saw the binomial theorem from a new perspective: linear algebra. At the time of his epiphany, Martin was a graduate student, preparing a lesson on the binomial theorem for his college algebra students. Martin's recent research had involved working with families of orthogonal polynomials, and he had spent a lot of time recently switching between bases for the vector space of univariate polynomials, $\mathbb{R}[x]$.

While he was preparing this lesson, Martin "had the familiar feeling of doing something [he] had been doing recently elsewhere," and he suddenly realized the binomial theorem could be thought of as "telling us how to expand a vector written in terms of the second basis as a linear [combination] of the vectors in the first basis." In other words, the binomial theorem could be thought of as a way to switch between bases of the vector space $\mathbb{R}[x]$, just as he had been doing in his research.

Martin gave the following example. The nonnegative integer powers of x (i.e., $1 = x^0, x, x^2, x^3, \ldots$) are a basis for the vector space $\mathbb{R}[x]$ of polynomials with coefficients in $\mathbb{R}$. Moreover, the nonnegative integer powers of $1 + x$ (i.e., $(1 + x)^0 = 1, (1 + x), (1 + x)^2, (1 + x)^3, \ldots$) constitute a different basis for the same vector space, $\mathbb{R}[x]$. This means that an equation like $(1 + x)^3 = \sum_{k=0}^{3} \binom{3}{k} x^k = 1 + 3x + 3x^2 + x^3$ can be thought of as taking the vector $(1 + x)^3$ written in terms of the second basis and rewriting it as the linear combination $1 + 3x + 3x^2 + x^3$ of vectors from the first basis. Martin reflected favorably on the experience: "Both things got simpler by virtue of their relation to each other. [I was] happy to have seen it, a bit surprised that I hadn't heard it said that way before." Moreover, because of the connection between Pascal's triangle and the coefficients of the binomial theorem, Martin realized that Pascal's triangle can be thought of as a change of basis matrix by simply writing the entries of Pascal's triangle in a lower triangular matrix (Figure 2).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 2: Pascal's triangle written as a change of basis matrix.

Theoretical Connections in the Literature

In our participants' narratives, we see that the factors identified by Hiebert and Grouws (2007) for the development of conceptual understanding may also promote backward transfer. We extend Perkins and Salomon's (1992) nomenclature of *mindful transfer* and *reflexive transfer* to the case of backward transfer. Furthermore, we argue that making connections explicit during instruction may promote mindful backward transfer.

We presented two narratives of instances in which individuals made connections between topics in advanced mathematics and the binomial theorem, a topic frequently taught in high school algebra classes. Neither of these individuals described any procedural change in the ways they operationalized the binomial theorem when solving problems, but these connections led to conceptual shifts in the ways they understood the binomial theorem. As such, these connections are examples of changes in what Hiebert and Grouws (2007) called *conceptual understanding*, as Robin and Martin developed new "mental connections among mathematical facts, procedures, and ideas." (Hiebert & Grouws, 2007, p. 380) as a result of their interactions with advanced mathematics – combinatorics and linear algebra, respectively.

In these two narratives, we also see examples of backward transfer: mindful backward transfer in Robin's case, and reflexive backward transfer in Martin's. Robin's connection between combinatorics and the binomial theorem originated in her desire to understand the connection and her own effort exerted while using the theorem in her combinatorics class. She had encountered the binomial theorem in high school and had reason to believe that such a connection existed due to the presence of combinations in the formula for binomial expansion and direct instruction on the connection to Pascal's triangle. However, her desire to understand *why* combinations were related to the idea of binomial expansion had caused her to consciously seek resolution. Martin's connection, however, was not tied to his effort exerted toward understanding the binomial theorem; rather, it came as a consequence of exerting effort on related tasks in his research. Even though he did not realize the tasks were related to the binomial theorem while he worked on them, his need to revisit the binomial theorem in his teaching forced him to interact with it, and the recency of his work on (unbeknownst to him) related mathematics left him primed to make the connection. While Martin was not actively searching for a connection, he was surprised to find one as he experienced "the familiar feeling of doing something [he] had been doing recently elsewhere."

These stories are indicative of the two key features Hiebert and Grouws (2007) identified for the promotion of conceptual understanding, namely, the need for teachers to attend explicitly to connections in instruction and the need for learners to struggle with the related mathematics. In the cases of Robin and Martin, their connections did not take place during direct instruction on the binomial theorem, but rather at a later time: either during a later mathematics course or while working on research and teaching. Yet, in Robin's case, explicit instruction had given her an awareness that she should search for a connection she had not yet made, thus making mindful backward transfer possible. And in each of these stories, we see that the person was actively engaged in learning advanced mathematics, yet the connections they realized were to mathematics they had encountered much earlier in their careers. Their struggles to understand *advanced* topics induced them to develop new understandings of more basic topics that aligned with these advanced ideas. Robin's struggle was directly related to the binomial theorem and her exploration of the role combinations played. In contrast, Martin's struggle was tangential: he did not describe a struggle with the binomial theorem, but rather a struggle with the content he related to the theorem, that of changing bases for the vector space of polynomials. Both instances of transfer resulted from focused engagement with the mathematics involved in the connections that were eventually formed.

These narratives point to two important implications for education and the training of future teachers. The first is that the formation of important mathematical ideas and connections between ideas often results from engagement with advanced mathematics. Each of these narratives tells a story about an individual who learned something new about a concept long after they were taught the concept. While neither of these individuals was prevented from performing the requisite tasks assigned in their coursework, and indeed may have already possessed some conceptual understanding of the binomial theorem, each of them developed a deeper understanding of how the binomial theorem related to other concepts through participation in advanced mathematics. If one of our goals as educators is that our students should possess rich conceptual understanding, these stories suggest that, even in cases where connections are made explicit, students may benefit from engagement with advanced mathematics.

Questions are often raised about why prospective schoolteachers are required to take such advanced mathematics courses as combinatorics and linear algebra if the reality is that they will never teach the advanced topics contained in those courses. The second implication of our analysis is that these advanced courses provide learners with an environment in which they must see basic mathematical ideas from a new perspective through engagement in productive struggle. Each of these stories tells us about how moving on to more advanced mathematics allowed these individuals to see how the basic mathematical procedures they had learned in school fit into the larger context of mathematics. The benefit of advanced mathematics courses for teachers may not lie in the advanced mathematics, itself, but rather in the advanced perspectives that teachers develop about the fundamental mathematics they will one day teach to others.

References

Ausubel, D. P. (1963). *The psychology of meaningful verbal learning*. Grune & Stratton.

Barnett, S. M., & Ceci, S. J. (2002). When and where do we apply what we learn? A taxonomy for tar transfer. *Psychological Bulletin*, 128(4), 612-637.

Brophy, J. E. (1997). Effective instruction. In H. J. Walberg & G. D. Haertel (Eds.), *Psychology and Educational Practice*. McCutchan.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404).

Hohensee, C. (2011). Backward transfer: How mathematical understanding changes as one builds upon it. [Doctoral Dissertation]. University of California, San Diego and San Diego State University.

Hohensee, C., Gartland, S., & Willoughby, L. (2019). Backward transfer effects on action and process views of functions. *Proceedings of The Forty-First Annual Meeting of the North American chapter of the International Group for the Psychology of Mathematics Education*, St. Louis, MO.

Murray, E., Baldinger, E., Wasserman, N., Broderick, S., & White, D. (2017). Connecting advanced and secondary mathematics. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1.

Perkins, D. N., & Salomon, G. (1992). Transfer of learning. In *International encyclopedia of education*.

Sala, G., & Gobet, F. (2017). Does far transfer exist? Negative evidence from chess, music, and working memory training. *Current Directions in Psychological Science*, 26(6), 515-520.

Wasserman, N. H. (Ed.). (2018). *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers*. Springer.

Weber, K., Mejia-Ramos, J. P., Fukawa-Connelly, T., & Wasserman, N. (2020). Connecting the learning of advanced mathematics with the yeaching of secondary mathematics: inverse functions, domain restrictions, and the arcsine function. *The Journal of Mathematical Behavior*, 57. https://doi.org/https://doi.org/10.1016/j.jmathb.2019.100752