Three Phases of Active Proof Reading Strategy for Comprehension of Mathematical Proofs: An exploratory study

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In advanced mathematics courses, proofs are instrumental in conveying mathematical knowledge. As a result, in many upper-division undergraduate math courses, proof comprehension is a crucial aspect of learning mathematics. This study examines proof reading strategies that eleven mathematics doctoral students profess to use to facilitate proof comprehension. This paper identifies 12 strategies that participants in this study employed to improve proof comprehension. I argue that those 12 proof reading strategies occur in three phases. In particular, doctoral students in this study engaged in three types of reading: preliminary reading, engaged reading, and reflective reading. The paper concludes with a brief remark on this research's implication for teaching proofs in undergraduate mathematics.

Keywords: Proof, proof comprehension, proof reading, undergraduate mathematics education

Students are expected to spend significant time reading proofs and glean mathematical knowledge from them in many upper-division mathematics courses. But recent studies show that proof comprehension is a notoriously difficult task for many students (Author, 2018). One way to support students' proof comprehension is to study how experts such as math professors read proofs. In an exploratory study, Weber and Mejia-Ramos (2011) identified three strategies research mathematicians generally employ when reading proofs: appealing to the authority of other mathematicians who read the proof, line-by-line reading, and modular reading. Moreover, in all three strategies, the authors noted that mathematicians used non-deductive reasoning. In another proof reading study, Weber and Mejia-Ramos (2014) surveyed 118 mathematicians and confirmed their earlier exploratory study findings. By modifying the research methodology used in Weber and Mejia-Ramos' (2011) study, this study seeks to further proof reading strategies that can facilitate proof comprehension.

Theory

Research on proof comprehension is continuing to uncover significant differences in proof reading strategies between novice (e.g., beginning undergraduate students) and expert (e.g., mathematics doctoral students) students. For example, Inglis and Alcock (2012) observed their participants' eye movement while reading a proof and concluded that undergraduate students, compared to the experts in their study, spend more time focusing on the "surface features" of a mathematical proof. Based on this observation, the researchers found that undergraduates spend less time focusing on proof's "big picture," such as the proof's logical structure. Inglis and Alcock's (2012) finding may explain why students often have difficulty understanding the logical structure of a mathematical argument, as evidenced elsewhere in the literature (e.g., Selden & Selden, 2003).

There is a growing body of research looking into strategies that can enhance comprehension of mathematical proofs. For instance, Weber and Mejia-Ramos' (2013) describes five proof reading strategies that undergraduates can use to facilitate their understanding. Their study observed four mathematics majors reading six proofs. The authors considered these students to be strong because they were successful in their content-based mathematics courses and on the follow-up proof comprehension test that the authors designed based on Mejia-Ramos et al.'s

(2012) proof comprehension assessment model. Their analysis revealed five proof reading strategies that the students used to facilitate their understanding of the proofs, which are (a) trying to prove a theorem before reading its proof, (b) comparing the assumptions and conclusions in the proof with the proof technique being used, (c) breaking a longer proof into parts or sub-proofs, (d) comparing the proof approach to one's own approach, and (e) using an example to understand a confusing inference. Weber and Mejia-Ramos (2013) followed up their qualitative study with a large-scale internet-based survey study that included mathematics majors and mathematicians from 50 large state universities in the United States. Their quantitative study's purpose was two-fold: (1) to explore whether mathematicians prefer mathematics majors to use these five proof reading strategies and (2) to investigate to what extent mathematics majors use these strategies. Their study's main finding is that the majority of mathematics majors do not use these proof reading strategies that mathematicians find useful for proof comprehension. This finding sheds light on why undergraduate students often gain little from proofs they read or see during lectures (e.g., Conradie & Frith, 2000; Rowland, 2001; Lew et al., 2016).

Research Methodology

Recall that this study's primary goal is to gain insight into how math Ph.D. students read proofs for understanding. The data to address the research question is obtained from interviewing eleven doctoral students in mathematics. I met with participants individually for a videotaped, semi-structured, hour-long, task-based interview. During the interview, participants were observed reading proofs of four distinct theorems. The content of three out of the four proofs is undergraduate mathematics; the other is from graduate-level mathematics. These interviews aimed to observe the strategies participants employed while reading proofs and examine how these strategies could contribute to proof comprehension. During the interview, participants were told that they could highlight or write on the proofs' text. Additionally, they were encouraged to think out loud while reading each proof.

After a participant completed reading the proofs, I asked several questions to explore their proof reading strategies. When participants said something I found confusing or interesting, I asked for clarification. The following questions are representative of my interview protocol:

- What are some of the things you were doing when reading these proofs to increase your understanding of them?
- Can you describe things you do to understand a proof? How do you know when you have understood a proof?
- When reading proof from a textbook or a journal article, what do you do when you get stuck?

When a participant did not mention any of the proof reading strategies described in Weber and Mejia-Ramos' (2013) study, I explicitly asked them how often, if any, they employed these strategies.

Data Analysis

Each interview was transcribed fully, and pseudonyms were assigned to the participants. Both the interview transcripts and analytic memos were studied carefully. The interview transcripts were initially analyzed using thematic qualitative text analysis (Kuckartz & Kuckartz, 2002; Maxwell, 2013). In the first pass through the data, any episode when a participant discussed something relevant to proof reading either prompted or otherwise was labeled as *a proof reading* strategy.

After identifying instances where participants talked about their proof reading strategies, I coded each instance using a semi open-coding scheme. Existing theoretical constructs from the literature on proof reading strategies such as Weber and Mejia-Ramos' (2013) study were used in the categorization process when I was confident that the literature's constructs agreed with the data. In contrast, when participants' responses were considered new or cannot be captured by existing constructs, I used *in vivo coding*—where terms used by participants are used as codes—and *process coding* to capture "observable and conceptual action in the data" (Saldana, 2013). I then organized similar codes into categories to develop a coding scheme, which was eventually checked by two experienced researchers, each with a Ph.D. in mathematics education and extensive research experience.

Finally, all interview transcripts were carefully examined and coded explicitly using the proof reading strategies identified in the coding scheme. Participants enacted behavior when reading a proof was cross-examined against a specific issue they were trying to address to facilitate their comprehension of the proof. For example, when I saw a participant underline a particular assertion or a definition in a proof or constructed a small sub-proof, I asked a follow-up question to understand their rationale for doing so.

Results

A careful analysis of the data suggests that participants in this study viewed mathematical proof reading as a non-trivial cognitive task that involves active engagement with the proof. Active engagement with proof is understood as any cognitive activity carried out to understand the proof. Taken together with previous literature on proof reading strategies (cf. Weber & Mejia-Ramos, 2011; Weber, 2015; Weber & Mejia-Ramos, 2013), the data in this study suggests that doctoral students engaged in three phases of reading cyclically: *preliminary*, *engaged*, and *reflective*.

Preliminary Reading

Preliminary reading is a theoretical construct that refers to a collection of strategies participants reported to employ to ascertain the *proof's framework* and overarching methods. Selden and Selden (1995) define *proof framework* as "the top-level structure of the proof, which does not depend on detailed knowledge of the relevant concepts" (p. 129). A key feature of *preliminary reading* is that participants are not interested in figuring out the proof's details; for instance, verifying the validity of an assertion or computation is less critical during *preliminary reading.* Instead, at this stage, the reader is interested in figuring out the overarching or "big picture" and motivating ideas of the proof. *Preliminary reading* can be considered an elaboration of what Weber and Mejia-Ramos (2011) consider *modular reading.* Table 1 below describes reading strategies participants may use during this stage of the active proof reading cycle.

Table 1

Table 1. Summary of Sub-categories or strategies included in preliminary reading, description of the content of the strategies, and the number of participants who mentioned them

Strategy	Content	Num

1. Understanding the statement of the theorem	Trying to understand the theorem by explicitly attending to concepts, objects stated or implied by the theorem (Weber, 2015)	3
2. Proving or sketching a proof for the theorem before reading its proof	Trying to construct a rough sketch or a complete proof of the theorem before seeing its proof (Weber, 2015)	10
3. Identifying the proof framework	Trying to understand how the proof proceeds by attending to what sub-claims are being proven, what proof technique is being used, what can be assumed, and what must be shown using that proof technique (Weber, 2015)	5
4. Skimming	Moving from one assertion to another without justifying them but to understand what is going on and how ideas in the proof are tied together to prove the theorem (Weber & Mejia-Ramos, 2011)	9

As shown in Table 1, the most frequently mentioned *preliminary reading* strategy is proving or sketching a proof for the theorem before reading its proof. Ten out of eleven participants said they used this strategy to enhance proof comprehension. For example, when I asked G7 if he tried to prove any of the theorems in the interview tasks before reading its proof, he responded:

> Some of them. I've seen [task] C before. [Task] B I did not try to prove it. Um [task] A I did try to prove it on my own and then [task] D I tried to prove it a little while, I thought about what is needed to be shown…I came up with one approach that was perhaps plausible and then I went to see what these authors had done. But I at least thought about it minimally. I think about what needs to be shown…

When I asked G7 how often he uses this strategy, he said:

> So you know ideally, if I were never in a rush, you know first I would try to prove it on my own and then um I want to make at least sure I know what needs to be shown and try to come up with some approach that seems like something that could work and then I'll look at the first line of the proof. And then maybe there's a key idea in the first line and say okay, now you know perhaps I can prove it from there.

It is important to note that when G7 uses this strategy—proving or sketching a proof for the theorem before reading its proof—he does not necessarily come up with a correct or complete proof; instead, an attempt or partial proof or laying out what the "landscape" of a proof may look like suffices.

Another participant, G8, also said he tried to prove the theorems before reading their respective proofs during the interview. When I asked him how frequently he uses this strategy, he said: "When I see a theorem, when I see a proposition or a statement, and I'm familiar enough with the ideas, and I will try and sketch out how I would understand it. If I'm unfamiliar with the ideas, I typically use the proof as learning material."

Participants in this interview discussed factors that influence *preliminary reading*: time and familiarity. For instance, G7 stresses the importance of time when deciding whether to sketch a proof of the theorem before reading its proof. As expected, G7 said he won't embrace the aforementioned *preliminary reading* strategy when he is in a "rush" or doesn't have enough time to execute the strategy. G8, on the other hand, pointed out that he makes a judgment call as to whether or not to use the strategy based on his level of familiarity with the underlying concepts of the theorem.

Engaged Reading

Engaged reading is a theoretical construct that refers to the set of strategies participants employed when trying to make sense of specific assertion (s) or an argument within a proof. Table 2 below describes seven strategies participants used during this phase of the active proof reading cycle.

Table 2

Table 2. Summary of sub-categories or strategies included in engaged reading, description of the content of the strategies, and the number of participants who mentioned them

Strategy	Content	Num
1. Justifying some assertions	Trying to fill in gaps to understand why a specific assertion is correct (note: some gaps could be filled mentally without having to write anything)	8
2. Partitioning the proof into smaller, manageable pieces	Trying to organize long proofs by breaking them into their modular structure	7
3. Using specific examples to understand a problematic assertion or series of assertions	Trying to use illustrations or diagrams to understand a complicated or confusing assertion	4
4. Zooming in	Trying to understand specific ideas in proof by conducting a unit analysis	2

5. Identifying, translating, and interpreting confusing assertions, key ideas, or notation using one's own words	Attending to higher-level concepts of the proof; making a note of key ideas or techniques employed in the proof	11
6. Forward-looking reading (Hide-and-seek)	Trying to complete the proof's remaining arguments after attending to the proof's *key ideas* and how they proceed.	4
7**.** Resource utilization	Relying on a book or a colleague for an explanation of a given assertion (s) within a proof	8

Engaged reading is by far the most active phase of the reading cycle. During this phase, participants were often observed underlining, highlighting essential ideas, and working out details of important assertions they found confusing. For example, when I asked G7 about some of the text he emphasized, he said: "… sometimes I underline something when I see that it's going to be used again later. So in this proof, we compare two computations. So we have this product over here, and then we have this product over here, so it's a visual guide." Another participant, G8, said:

> In proof B, I only underlined one thing. I probably should have highlighted two. I should have emphasized abelian group, and every non-identity element is of order two, right? That's the key idea, right? We have an algebraic structure…we have a group structure in particular. And everything is of order two, which means that it's a 2-group.

It is important to note that G8, in particular, said he underlines what he considers to be the proof's *key ideas*.

Reflective Reading

Reflective reading is a theoretical construct that refers both to a specific strategy and an analytical main coding category. As a strategy, *reflective reading* relates to instances where participants were observed reflecting on the whole or part of the proof. Evidence of this strategy include: (a) summarizing the entire or part of the proof using one's own words, (b) reflecting on specific takeaways such as a proof technique or *key ideas* from reading the proof, and (c) frequently asking oneself what a particular assertion in the proof says about the mathematical object under exploration within the proof. *Reflective reading* was a popular strategy utilized to improve proof comprehension. Nine out of eleven participants said they frequently use this strategy to enhance proof comprehension. The excerpt below illustrates introspective habits G8 engages in while reading a proof.

> If I think my approach is more elegant than the author, I will, which is not typically the case, think about the author. I'll try and read into what the author was trying to present. I'll try, and I'll ask myself if I'm using techniques, if I misunderstand something. If my approach is significantly more elegant. Am I imagining that something is nicer than it is? Am I imagining that certain facts are stronger than they are? So, am I cheating? When that's not the case, I will ask myself what is the author trying to present? Are they setting

> me up for a corollary afterward that uses the ideas from this proof, which this particular approach makes more explicit? When the author's proof is better than something, I can sketch it out for myself. I will examine the methods, ideas that they used, and how those better encapsulate the ideas, better sum together, piece together the ideas that I'm trying to use. So were they able to assemble these landmarks a bit better? Were they able to connect them in a much more direct fashion?

Discussion and Concluding Remarks

This exploratory study is an attempt to support students' proof comprehension by carefully examining characteristics of proof reading strategies exhibited by experts, mathematics doctoral students. This study's key finding is that proof comprehension is an active cognitive process that involves several strategies that can be grouped into three reading phases: *preliminary*, *engaged*, and *reflective*. Typically, participants in this study would engage in a *preliminary reading* first and then proceed to the most active part of the reading phase: *engaged reading*. Typically, participants dive into *reflective reading* once they feel like they have an adequate understanding of the proof in its entirety, usually after *preliminary* and *engaged reading*. However, it could be the case that participants may elect to *reflect* on a small subset of assertions in the proof before understanding the whole proof. This study also revealed potentially effective proof reading strategies that were not documented in prior studies described earlier (e.g., *Identifying, translating, and interpreting confusing assertions or notation using one's own words*, *forward-looking reading*, and *resource utilization*).

Additionally, this research may shed light on a relationship between the proof comprehension assessment model proposed in Mejia-Ramos et al. (2012) and proof reading strategies. Recall that Mejia-Ramos et al. (2012) argue that proof comprehension can be assessed *locally* and *holistically*. A proof can be understood either locally by "studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof, or holistically based upon the ideas or methods that motivate the proof in its entirety." (p.6) For instance, several of the strategies participants employed during *engaged reading* phase could support *local* comprehension; whereas, strategies employed during *preliminary reading* and *reflective reading* may enhance *holistic* comprehension of a proof.

To sum up, the difficulty associated with proof comprehension can be assuaged by exposing students to strategies that can enhance proof comprehension. If future studies confirm that the proof reading strategies presented here are effective in improving proof comprehension, mathematicians could incorporate them into their teaching. For instance, professors who teach proof-based mathematics courses, such as introduction to mathematical proof courses, can explicitly communicate these strategies to their students. A professor could model some of the aforementioned strategies by incorporating proof reading activity during a lecture. Moreover, a professor can develop proof comprehension questions using the Mejia-Ramos et al. (2012) model and foster these strategies' usage.

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