Adapting a Single Local Instructional Theory into Multiple Instructional Sequences

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Local instructional theories (LITs) are often described as a generalized sequence of steps for students' guided reinvention of some mathematical concept. They consist of a series of tasks and the accompanying rationale for the tasks, where the rationale is described in student's mathematical activity. Literature suggests that the rationale can be used to adapt LIT's into different instructional sequences however it fails to provide specifics as to how these adaptations are achieved. This report details how the design heuristic of didactical phenomenology can be used to create multiple instructional sequences for students with different mathematical backgrounds, thus contributing to our understanding of the theory of realistic mathematics education.

Keywords: realistic mathematics education, didactical phenomenology, group theory

Abstract algebra is a required course for almost every undergraduate mathematics student (Blair, Kirkman, & Maxwell, 2013) and it is notoriously difficult for them (Larsen, 2010; Leron, Hazzan, & Zazkis, 1995; Weber & Larsen, 2008). In fact much of the literature on student understanding in abstract algebra highlights student struggles (Weber & Larsen, 2008). One of the many contributing factors to student difficulties in the subject is the abstract nature of the content of the course (Hazzan, 1999). In an attempt to help students navigate the abstract nature of the content, I recently engaged a variety of students in a real-world application of abstract algebra by investigating molecular structures. The findings reported here are from a larger design experiment aimed at developing a local instructional theory (Gravemeijer, 1998) for the guided reinvention of a classification system (i.e. flowchart) for chemically important symmetry groups (Bergman & French, 2019; Bergman, 2020).

As students begin to develop a system for identifying various symmetry groups, they need to decide on a group representation that is encapsulated, or compact, enough to be used as the outputs for their classification system. While there are a number of ways to represent the group concept (Bergman et al., 2015), one such as Cayley tables would be rather impractical to serve as the terminating points of a flowchart. Therefore, students were ultimately encouraged and supported in using something like a group name, i.e. $\mathbb{Z}_3$, D_8, $\mathbb{Z}_2 \text{ x } \mathbb{Z}_2$. This report gives evidence and explanation to how I used the design heuristic of didactical phenomenology to support students with different mathematical backgrounds in their use and reinvention of group names, thus attending to the research question: *What kinds of mathematical activity can promote the evolution of the students' informal knowledge and strategies into more powerful ways of thinking, symbolizing, and reasoning?*

Theoretical Framing

The design study reported here utilized the instructional design theory of realistic mathematics education, RME, in order to develop a local instructional theory, LIT (Gravemeijer, 1998). RME provided both an underlying theoretical perspective and three accompanying design heuristics. The notion that mathematics is a human activity instead of a collection of facts and procedures is the theoretical foundation of RME (Freudenthal, 1971). The first design heuristic of RME, the reinvention principle, suggests that by engaging students in the guided reinvention of a particular mathematical concept, we are creating opportunities for them to meaningfully engage in this activity of mathematizing. When used as a design heuristic, the reinvention principle promotes the idea of starting the reinvention process in a context that is experientially real to students (Gravemeijer & Doorman, 1999). The reinvention/evolution from an informally situated student strategy to conventionally recognizable, formal mathematics is evoked using the second design heuristic, emergent models. Emergent models help to describe the overall evolution of the students' mathematical activity. Since the formal mathematics that is eventually developed is rooted in the students' approaches, this particular development process helps students have a greater sense of ownership over the formal mathematics created. The students' use of the encapsulated group representation described in this report can be seen as the final version of the students' model for their unique symmetry group, which was eventually incorporated into their classification system.

The third and final design heuristic of RME is didactical phenomenology. Larsen (2018) states that, "Simply put, didactical phenomenology tells the designer that an instructional sequence meant to support the learning of a piece of mathematics should be situated in a context that can be productively organized by students using that piece of mathematics." (p.25) As a heuristic it is helpful to the design researcher in two distinct ways. First, didactical phenomenology is used to help determine an appropriate context for the students to engage with. For example, since the goal of the study was to support students in reinventing a classification system for the various symmetry groups of molecules, I began by having them first investigate the symmetry groups of specific molecules with a variety of group structures. The second way in which didactical phenomenology is useful to the design researcher is in helping the evolution of the student's mathematics from informal to more formal mathematics through progressive mathematizing. The specifics of how didactical phenomenology was used to promote this evolution in students' use of group representations are presented as the main results of this report.

Methods

The overarching design study consisted of three teaching experiments which were conducted with pairs of students with varying mathematical backgrounds. The data reported here is from the second and third teaching experiments exclusively. The second teaching experiment, TE2, involved a pair of senior level undergraduates, Arthur and Stu, who had recently completed a traditional proof-based introductory group theory course. The third teaching experiment, TE3, was conducted with Ada and Sophie, sophomores majoring in mathematics and engineering, respectively. Ada and Sophie had recently completed an introductory linear algebra course and neither had any previous experience with group theory or advanced mathematics.

The corpus of data included video recordings of the teaching experiment, all the written work which was scanned into pdf documents, and researcher memos created during ongoing analysis. TE2 consisted of 12 sessions lasting 60-90 minutes each, and TE3 was 11 sessions also lasting 60-90 minutes per session. Between sessions, memos (Maxwell, 2013) were used for recording: the goals for each session, a short recap of the students' mathematical activity after each session, and a description of how well the goals were met. During retrospective analysis content logs were used to record a description of the students' activity, highlight places the students moved forward in the LIT, and helped organized my understanding of the student's activity overall.

Results

The results are organized in two cases, each detailing a different instructional sequence created in response to the students' unique mathematical activity. Each case contains a description of the student's initial model of their symmetry groups and an account of the transition of this informal model into a model for more formal mathematical activity using the notions of didactical phenomenology.

Case 1: Students with prior group theory experience

For the students in TE2 who had previous exposure to conventional group theory concepts, their goal for representing their symmetry groups was to use "standard group names." By "standard name" I am referring to the paradigmatic representatives of the isomorphism classes to which each symmetry group belongs. For example, there is only one algebraic structure that satisfies all of the group axioms with three elements. This isomorphism class is often referred to as $\mathbb{Z}_3$ even though the elements in the set are often referring to something other than $\{0,1,2\}$ under modular addition. It is common practice in abstract algebra, and also in chemistry, to use a single name to describe an entire isomorphism class. In chemistry, they do it quite explicitly by saying ammonia belongs to D_6.

The mathematics offered by the students as a starting point. When asked to identify the symmetry groups of water, ammonia, and ethane (in an eclipsed configuration) students in TE2 seemed to have some access to standard group names. Evidence of student's mathematical activity naming the symmetry group of ethane is given below.

Stu: I think we could rep… like one way I might wanna think about starting is um, we know that we could compare it to this (ammonia) because this half of ethane looks an awful lot like ammonia, right? (see Figure 1 below)

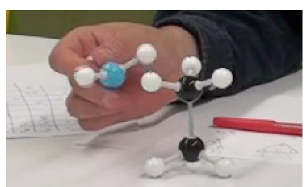


Figure1. Stu's visual comparison of ammonia and ethane

Arthur: yeah

Stu: So I feel like it might be something like $\mathbb{Z}_2$ cross ammonia. If that makes any sense?

How the student's progressive mathematization was supported using didactical phenomenology. Referring to the symmetry group of ethane as "$\mathbb{Z}_2$ cross ammonia" is mathematically inaccurate for a number of reasons; ammonia isn't technically a group, none of the elements in the symmetries of ethane are integers under modulo addition, and there is no external direct product between two groups. However, Stu's suggestion was also quite promising for a number of reasons as well. The symmetry group of ethane, which would conventionally be considered an internal direct product, is isomorphic to the external direct product D_6 x $\mathbb{Z}_2$ and the students were already comfortable with the symmetries of ammonia as they had recognized it as equivalent to a more familiar group from their introductory group theory course, the group of symmetries of a triangle. This posed the instructional design question: How can students be supported in connecting their intuitive notions of different group structures with conventional representations?

Where's the D_6 and where's the $\mathbb{Z}_2$? To better understand the students' use of a direct product, I asked them to explain how they were seeing the groups D_6 and $\mathbb{Z}_2$ in their group of symmetries of ethane. Note that after Stu's previous suggestion of "$\mathbb{Z}_2$ cross ammonia" we discussed that the symmetry group of ammonia could also be referred to by the more conventional name of D_6.

Researcher: Last time you said that the symmetry group of ethane was D_6 x $\mathbb{Z}_2$ can you tell me, what's the D_6 and what's the $\mathbb{Z}_2$? Like where is D_6 and where is $\mathbb{Z}_2$?

Arthur: D_6 was the configuration of the three kind of top hydrogen as we were looking at them and $\mathbb{Z}_2$ encoded which face was up. Or in the case of these colorings, just cause we needed a way to tell the two triangular faces apart. So it was a case of the six combinations and then like a binary dimension to that, which is what crossing $\mathbb{Z}_2$ encoded.

This description of their symmetry group is encouraging. Their portioning of the set into two subsets is powerful since the group D_6 x $\mathbb{Z}_2$ can easily be partitioned into two copies of D_6 where one has a 0 in all the second components and the other has a 1 in all of them. By describing all the symmetries as the product of something in D_6 and "either color-swapped or not" they seem to be capturing the notion that all the elements in the group can be represented as some hk for two subgroups H and K, which is a necessary condition of internal semi-direct products.

Prove that your symmetry group for ethane is isomorphic to a more conventional representation of D_6 x $\mathbb{Z}_2$. In order for Arthur and Stu to meaningfully connect their intuitive notions of group structures to more conventional representations, they spent a lot of time in TE2 trying to prove that their symmetries of ethane were isomorphic to D_6 x $\mathbb{Z}_2$. A more detailed discussion of how the students' mathematical activity related to isomorphisms was supported using didactical phenomenology is beyond the scope of this conference report.

How do you know you have a D-something x $\mathbb{Z}_2$? What would you look for? After having spent a lot of time focused on various representations of D_6 x $\mathbb{Z}_2$ and making arguments about it's group structure, the students were asked to consider a similar group structure in a more generic context.

Reseracher: If I just handed you another molecule could you tell if it was going to be a *D*-something cross $\mathbb{Z}_2$? What would you look for? Describe something that you would expect to have a *D*-something cross $\mathbb{Z}_2$ symmetry group.

Arthur: I'd look for that regular polygon. I would look to be able to see it, how do I say this… looking for regular polygon, I feel good about that part of the process, and then I guess saying like how many nested symmetries now, because I can look at this square and there's another square that I can look at that's indistinguishable from the first. So that's kind of our $\mathbb{Z}$-category, our number of color options.

Researcher: …and then the $\mathbb{Z}_2$, sorry one more time…

Stu: Is coming from this plane of reflection, between the two polygons. And it's not really the molecules it's the plane of reflection.

The goal for this task was to test the students' familiarity with identifying groups that are isomorphic to external direct products and to see if they had begun to gain more ownership over the use of conventional group names. Stu's understanding that a plane of reflection is necessary to pair with a regular polygon is particularly encouraging. Stu's attention to an independent symmetry can be interpreted as evidence of him being at least implicitly aware of the importance of the commutativity of r_h (the horizontal reflection associated with their "color swap") with the other symmetries. This symmetry is independent as in it doesn't have an impact on the orientation, which again is a function of commutativity. What Stu is seeing is important because it is the consequence of this commutativity, and commutativity establishes the equivalence between their internal direct product and the external direct product they used to describe their symmetries. While a strong argument can be made that Stu is observing the necessary conditions for an internal direct product to be met, it is also clear that he faces an obstacle in using the formal language necessary to describe why what he is observing is the necessary condition for this to have the proper group structure.

Summary of Case 1. Arthur and Stu start with a representation of a symmetry group that was based on their situated activity with ammonia and their previous exposure to external direct products. This informal activity was the motivation for a didactical phenomenological analysis aimed at supporting students in the use of more formal group names in a new more general mathematical reality. In other words, the students had gone from an informal notion of what a direct product represents to being able to describe what kinds of molecules will have this particular kind of group structure and why.

Case 2: Students without prior group theory experience

For the students in TE3 who had no previous exposure to group theory concepts, my goal for them was to reinvent a compact representation *like* group names to use in their flowchart since conventional group names were unavailable to them. Early in the teaching while establishing geometric symmetries as an experientially real starting point, Ada and Sophie were guided in reinviting the group concept by investigating the symmetries of square. This activity led them to describe their groups using what are called group presentations. Group presentations are comprised of a set of generators, and the relations between them.

The mathematics offered by the students as a starting point. When asked to identify the symmetry groups of water, ammonia, and ethane (in an eclipsed configuration) students in TE3 created a group presentation for each as seen in Table 1 below.

Table 1. Student's group presentations for the symmetry groups of water, ammonia, and ethane.

Water	Ammonia	Ethane (eclipsed)
$\mathbb{Z}_2$ x $\mathbb{Z}_2$	D_6	D_6 x $\mathbb{Z}_2$
0F=F 0R=R 2R=0 2F=0 Associative FR=RF Inverses	$R^3=0$ $F^2=0$ 0R=R 0F=F $RF=FR^3$	RF=FR $T^2=0$ $R^3=0$ $F^2=0$ FR=RF $RT=TR^2$ TF=FT

How the student's progressive mathematization was supported using didactical phenomenology. While this representation of a group is useful for capturing the structure of the group, the students needed a more concise representation to serve as the output for their classification system. In this case the student's mathematical activity posed the instructional design question: How can students be supported in transitioning from the use of group presentation to a more compact representation like a group name? Since conventional group names were unavailable to their students due to their lack of previous exposure to group theory, the question became: How can students be supported in the reinvention of group names? Fortunately, the students' mathematical activity of creating group presentations provided a context in which to seek solutions to these questions by conducting phenomenological analyses.

Test out your system on a new set of molecules. Ada and Sophie were given a new set of molecules and asked to determine the symmetry group for each, as seen in Table 2 below. This new set of molecules were specifically chosen to give the students an opportunity to work with each of the most common group structures found in chemically important symmetry groups: cyclic, dihedral, cyclic x $\mathbb{Z}_2$, and dihedral x $\mathbb{Z}_2$. It was necessary to expose the students to each of the group structures so that they would have sufficient names for each group in their flowchart. When given the new molecules, the students quickly and efficiently produced group presentations, seen in Table 2.

Table 2. Student's group presentations for the symmetries of a new and expanded set of molecules.

Tetra-aza copper II	Hydrogen peroxide	Boric acid	BrF_5	cyclobutene
$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_3$ x $\mathbb{Z}_2$	D_8	D_8 x $\mathbb{Z}_2$
TAC $R^4=0$ 4 symmetries No Fs No Ts	Hydrogen Peroxide $R^2=0$	Boric Acid $R^3=0$ $T^2=0$ 6 symmetries $TR=RT$	Brain Slug 4 rotations $R^4=0$ $F^2=0$ 8 symmetries $RF=\cancel{F}FR^3$	Cyclobutane $R^4=0$ $F^2=0$ $T^2=0$ 16 symmetries $RF=FR^3$ $TR=RT$ $TF=FT$

Make a 'score card' of written rules for each molecule. Once the students had exposure to each of the different flavors of symmetry groups, Ada and Sophie were asked to make "score cards" for each of the molecules they had encountered so far containing each of their group presentations. Creating the score cards involved going though previous work to find each of the group presentations; doing so gave Ada and Sophie an opportunity to both standardize their notation and streamline their presentations. This standardization can also be interpreted as a formalization of their mathematics and therefore evidence of progressive mathematization.

Sort the score cards into collections/flavors/kinds of symmetry groups. Which seem more similar? Which hangout together? When asked to sort the score cards into collections of similar groups types Ada and Sophies initially offered two different strategies. Ada started by gathering the pure rotational groups together, whereas Sophie gathered the groups with order 3 rotations. Sophie speaks first suggesting that they put all the groups with the same order rotation together, but then quickly stops saying:

Sophie: Oh wait, what about the F's and the T's?

Ada: I kinda inherently wanna be like, oh there are ones that have R's and F's , R's and T's, R's, T's, and F's, or just R's.

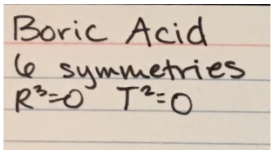
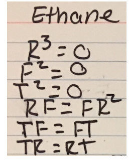
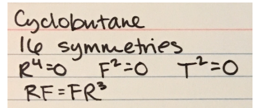
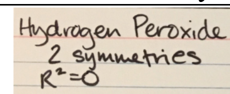
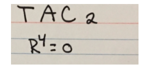
Sophie: Yeah, it seems, it's totally grouping them.

Ada: I think that would be like my gut instinct.

They quickly work together to arrange the groups into four collections based on the existence of symmetries within the group (seen in Table 3). This categorization was extremely exciting because not only were the students one step closer to using a more useful group representation for their flowchart, this is precisely how group structures are conventionally categorized in the context of chemistry.

Table 3. Student's sorting of group presentation by the existence of symmetries.

Contain's *R* and *F*		
Water 4 symmetries $R^2=0$ $F^2=0$ $RF=FR$	Ammonia $R^3=0$ $F^2=0$ $\cancel{RF=FR^2}$ $RF=FR^{n-1}$	BRF^5 (Brain Slug) 8 symmetries $R^4=0$ $F^2=0$ $RF=FR^3$

Contains R and T	Contains R, T, F	Contains only R's
	 	 

Come up with a name for each of the collections. Once Ada and Sophie were comfortable with their grouping of the groups into four categories, they were asked to come up with a "name" each of their categories of groups. The students quickly agreed on naming each collection of groups based on the symmetries that exist within the groups. Therefore, they ultimately created 4 symmetry group types; *RT group*, *RF group*, *R group*, and *RFT group*.

Summary of Case 2. In order to support Ada and Sophie in describing symmetry groups with concise group representations, the students were provided with supplemental tasks and prompts to support their progressive mathematizing. The sequence of tasks used to help the students move from group presentations to group names gives evidence to how the students' informal mathematical activity was leveraged into more formal mathematics.

Discussion and Conclusion

My goal has been to show how didactical phenomenology can be used to create multiple instructional sequences for students with different mathematical backgrounds while engaging in the same LIT. The pairs of students in the two teaching experiments had access to two very different sets of mathematical experiences. For students in TE2 their previous exposure to group theory led them to focus on the use of conventional group names. However, the students had only an informal notion of the group structures needed to describe molecular structures. For students in TE3 their lack of access to conventional groups granted them an opportunity for reinvention. Didactical phenomenology was used in both cases to create tasks that could promote the transition of the student's informal ideas into models-for more formal mathematical activity. By attending to aspects of the students' mathematical activity that begged to be further mathematized by the use of a group name, individualized instructional sequences were created to support students in their reinventing and use of group names.

Implications

The implications from these results are twofold. First, this work adds to our collective understanding of the theory of realistic mathematics education. Literature describing LIT's states that the sequence of tasks included in a LIT is meant to be general in the sense that it can be adapted to work in different instructional sequences (Larsen, 2018). However, when LIT's are presented they are often described in terms of the mathematical activity of students with similar mathematical backgrounds (Larsen, 2009; Larsen & Lockwood, 2013; Swinyard, 2011; Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013). This work begins to shed light on how exactly these adaptations can be made. Second, this work starts to push on the notion of pre-requisites and access in advanced mathematics. Access to advanced math nearly universally comes after a long path of calculus and differential equations with no breaks. This is a sequence of opportunities to get kicked out with no alternative ways to get back, where we use surviving as a gatekeeper. The instructional sequence resulting from TE3 explicitly provides a way for students to engage in meaningful advanced math without much previous experience as a pre-requisite where they can be supported in building formal mathematics off their own informal activity.

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