EXTENDING A LOCAL INSTRUCTIONAL THEORY FOR THE DEVELOPMENT OF NUMBER SENSE TO RATIONAL NUMBER

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We report on the process by which we extended a local instruction theory for number sense development from the whole-number domain to the rational-number domain. Students involved in an earlier teaching experiment developed improved number sense, particularly in the form of flexible mental computation. The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory. The present study concerns a recent classroom teaching experiment in which envisioned learning routes that were developed in the context of whole-number mental computation and estimation were applied to reasoning about fraction size. In this way, the application of the local instruction theory was extended from whole-number sense to rational-number sense.

Keywords: Local instruction theory, number sense, prospective teachers, rational number

Gravemeijer (1999, 2004) introduced the construct of *local instruction theory* (LIT) and has encouraged the development of LITs to support reform efforts in mathematics education. A *local instruction theory* consists of "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). We have written about our LIT for number sense development (Nickerson & Whitacre, 2010), which informed instruction in a previous classroom teaching experiment (Cobb, 2000). Our LIT was refined on the basis of the results of that teaching experiment. In our previous work, the focus was on developing number sense in prospective elementary teachers within the whole-number domain, and the particular focus was on mental computation and computational estimation. Our recent research involved extending the LIT for number sense development to the rational-number domain, with a focus on students' reasoning about fraction magnitude.

Background

This study represents the latest phase in an ongoing design research effort concerning the number sense development of prospective elementary teachers. In this section, we the stage for the present study with a brief review of relevant literature and a description of our approach to instructional design.

In our work, we conduct design research in the form of classroom teaching experiments, which are reflexively related to theory building (Cobb & Bowers, 1999). Design research occurs in cycles of instructional design, classroom teaching experiments, and data analysis (Gravemeijer, 1999). We see these cycles occurring at two distinct grain sizes: (1) The cycles are classroom teaching experiments, which are informed by and inform the LIT; (2) Within a given classroom teaching experiment, the LIT informs the design of hypothetical learning trajectories (Simon, 1995). A hypothetical learning trajectory (HLT) is a vehicle for planning the learning of particular mathematical topics. An HLT consists of the goal for students' learning, the mathematical tasks, and hypotheses about the process of students' learning.

Note that a local instruction theory is distinct from a hypothetical learning trajectory in two ways: (1) an LIT tends to describe an instructional sequence of longer duration; (2) an HLT is situated in a particular classroom, whereas an LIT is not (Gravemeijer, 1999). An LIT for the development of number sense would need to be generalizable to underlie many mathematical topics. Local instruction theories are an important part of our work as mathematics teacher educators teaching prospective elementary school teachers.

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply (Ball, 1990). However, prospective and practicing elementary teachers often know the procedures of elementary mathematics, but do not understand the material conceptually (Ball, 1990; Ma, 1999). In fact, studies of preservice elementary teachers have found that this population tends to exhibit poor number sense, *even after* having completed their required college mathematics courses (Tsao, 2005; Yang, 2007; Yang, Reys, & Reys, 2009). In light of these findings, our research has focused on activities associated with improving the number sense of prospective elementary teachers.

Number Sense

One major area of focus in the number sense literature has been an investigation of the computational strategies that students use. Good number sense is associated with flexibility, which is exhibited in the use of a variety of computational strategies (Heirdsfield & Cooper, 2004). This is in contrast to individuals who exhibit little flexibility. In mental computation in older children and adults, inflexibility often manifests in the use of the mental analogues of the standard paper-and-pencil algorithms. While the standard algorithms can be useful, skilled mental calculators tend to select a strategy for an operation based on the particular numbers at hand. Furthermore, the strategies that these individuals use often stray far from standard, as in reformulating computations or rounding and compensating (Carraher, Carraher, & Schlieman, 1987; Greeno, 1991; Hierdsfield & Cooper, 2002; 2004; Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys, Rybolt, Bestgen, & Wyatt, 1982; Reys, Reys, Nohda, & Emori, 1995; Sowder, 1992; Yang, Reys, & Reys, 2009). The literature also suggests that these strategies should develop in an environment in which students are encouraged to develop their own strategies, share these strategies, and reflectively and collectively discuss the merits of different approaches (McIntosh, 1998; Sowder, 1992).

In 2005, we conducted a classroom teaching experiment in a mathematics content course for prospective elementary teachers. This course was the first in a sequence of four content courses and had a focus on number and operations. In that study, we focused on mental computation as a microcosm of number sense. Instruction was guided by our local instruction theory for the development of number sense. Thirteen students participated in pre/post interviews in which they were given story problems to be solved mentally. In addition to coding for the particular strategies that participants used, we coded these as belonging to more general categories of strategies indicating the extent to which the person's approach is tied to (or departs from) the standard algorithm for a given operation. We used a scheme of Markovits and Sowder (1994) to categorize mental computation strategies as *Standard*, *Transition*, *Nonstandard*, and *Nonstandard with Reformulation*. Nonstandard strategies are those that diverge substantially from the standard algorithms. The use of such strategies suggests an understanding of the operation that is not bound to any particular algorithm. As such, nonstandard strategies are associated with number sense (Markovits & Sowder, 1994; Yang, Reys, & Reys, 2009).

When the results were reviewed with respect to the Standard-to-Nonstandard framework, it was evident there was a rather dramatic shift in the strategies used by the 13 interview participants. In the first interview, Standard strategies were most common. Participants used few strategies for the operations, and often relied on the mental analogues of the standard written algorithms. In the second interview, by contrast, the most common category of strategy was Nonstandard with Reformulation – at the other extreme of the continuum. Thus, participants shifted from using the most standard to the least standard strategies, which suggests that their understanding of the operations moved from being bound to the standard algorithms to being unconstrained by these (Whitacre, 2007). These results were encouraging and led us to pursue further research concerning the number sense development of prospective elementary teachers.

LIT for the Development of Number Sense

Our local instruction theory for the development of number sense is organized around three major goals: (1) Students capitalize on opportunities to use number-sensible strategies; (2) Students develop a repertoire of number-sensible strategies; (3) Students develop the ability to reason with models. The third of our goals was influenced by one of the design heuristics of Realistic Mathematics Education (RME): students and teachers develop models *of* their informal activity, which become models *for* more formal mathematical reasoning (Gravemeijer, 1999; 2004; Stephan, Bowers, Cobb, & Gravemeijer, 2003). In this short paper, we focus on Goals 1 and 2. Table 1 describes the instructional activities and corresponding envisioned learning route for Goal 1. We think about the route to Goal 1 as a progression in which students move from being reliant on the standard algorithms toward understanding numbers and operations independently of those algorithms. In a class in which number sense development is a major goal, students initially are not accustomed to exercising their number sense when opportunities arise. The instructional activities in Table 1 are intended to encourage students to recognize and take advantage of those opportunities.

Table 1. Route to Goal 1.

Instructional Activities	Envisioned Learning Route
Instructor identifies and engineers opportunities for computational reasoning	
Students are invited to engage in computational and quantitative reasoning	Many students initially rely on standard algorithms
Students are encouraged to carry sense making to solutions with nonstandard strategies	Students use their own nonstandard strategies
Students engage in computational reasoning in a variety of contexts	Students capitalize on opportunities to use number-sensible strategies

Table 2 describes the instructional activities and corresponding envisioned learning route for Goal 2. This progression focuses on students' explicit awareness of strategies through discourse concerning these. We envision students moving from thinking/talking about how they solved a particular task to looking/discussing strategies that they and their peers have used across a set of tasks. As students reflect on these strategies and make them objects of discourse, they develop an explicit repertoire of strategies that they can draw from, depending on the details of the task at hand.

Table 2. Route to Goal 2.

Instructional Activities	Envisioned Learning Route
Instructor anticipates the nonstandard strategies that students might use	

Class negotiates records of strategies and initiates the practice of naming	Students may initially name strategies in ways tied to specific examples
Instructor and students negotiate differences and relative efficacy of strategies	Students name strategies according to essential characteristics
Instructor and students make strategies objects of discourse	Students develop a repertoire of number-sensible strategies

Note that these tables differ somewhat from those presented in Nickerson and Whitacre (2010) in terms of both format and content. The progressions remain very much the same, but we have modified our language in order to broaden the scope of the LIT. We use the term *computational reasoning* to encompass various activities that involve reasoning about numbers and operations and that have been associated with number sense. Among these are: mental computation, computational estimation, and reasoning about fraction magnitude.

Applying the LIT to Reasoning about Fraction Magnitude

Originally, our LIT was developed with a focus on whole-number mental computation. In the 2010 study, we sought to extend it to the rational number domain, and we chose to focus on students' reasoning about fraction magnitude. This area is related to mental computation in that it too consists of tasks for which there are a variety of strategies that can be used. There are traditional procedures, which are often taught in school, as well as various nonstandard strategies. Behr, Wachsmuth, Post, and Lesh (1984) touted the importance of reasoning about fraction magnitude as a prerequisite to reasoning meaningfully about operations involving fractions. Furthermore, fraction estimation and comparison tasks have been used in assessments of students' number sense (Hsu, Yang, & Li, 2001; Reys & Yang, 1998; Yang, 2007).

A framework of Smith (1995) informs our thinking concerning reasoning about fraction magnitude. Smith groups fraction comparison strategies into four categories, which he calls *perspectives*. These perspectives serve not only to categorize strategies but also to highlight commonalities in reasoning across groups of strategies. Strategies such as converting to a common denominator or converting to a decimal belong to the *Transform* perspective. They involve transforming one or both fractions in some way in order to facilitate the comparison. One can also compare fractions without performing any sort of transformation. One way to do this is to apply the *Parts* perspective, wherein the fractions are interpreted in terms of parts of a whole. This perspective alone is sufficient for relatively simple cases, such as comparing fractions that have the same numerator or same denominator.

The *Reference Point* perspective involves reasoning about fraction size on the basis of proximity to reference numbers, or *benchmarks* (Parker & Leinhardt, 1995). For example, using the *residual strategy*[15] for comparing fractions, one compares the difference of each fraction from a common benchmark number, typically 1: To compare 7/8 an 6/7, we can notice that 7/8 is 1/8 away from 1, whereas 6/7 is 1/7 away from 1. Since 1/8 is less than 1/7, 7/8 is closer to 1, and therefore larger (Yang, 2007). The *Components* perspective involves making comparisons within or between two fractions, as in coordinating multiplicative comparisons of numerators and denominators. For example, in order to compare 13/60 and 3/16, we can notice that $13 \times 5 = 65 > 60$, whereas $3 \times 5 = 15 < 16$. It follows that 13/60 is greater since its numerator-denominator ratio is less extreme.

[15] Smith refers to this as the *reference point* strategy, as do Behr, et al., 1984.

In designing instruction, these perspectives informed our decisions relative to tasks, number choices, and anticipated student reasoning. We mapped out the envisioned learning routes described in our LIT in terms of the evolution of these perspectives and of particular strategies within each category.

Although Smith (1995) does not describe the perspectives or particular strategies belonging to his framework in a hierarchical way, we view the Reference Point and Components perspectives as generally more sophisticated categories of reasoning about fraction size.[1] There is support for this in the literature. For example, Yang (2007) considers the residual strategy to be Number sense-based, as opposed to Rule-based. We posit that there is a general correspondence between Smith's perspectives and the Standard-to-Nonstandard framework, described earlier. In particular, the Transform and Parts perspectives correspond more or less to the Standard and Transition categories of strategies, while the Reference Point and Components perspectives correspond to Nonstandard strategies (with or without reformulation). We do not intend by this a one-to-one mapping of categories, but a more general grouping into Standard (including Transition) and Nonstandard. Smith distinguishes between *instructed* and *constructed* strategies for comparing fractions, based on the degree to which he found support for these in popular textbooks. He found little support for Reference Point or Components strategies in texts. Smith's constructs of instructed and constructed marry well with the categories of Standard and Nonstandard.

Based on our previous teaching experience, together with a review of the literature, we expected that the prospective elementary teachers would come to our course with limited number sense and would tend to apply standard algorithms for comparing fractions (Newton, 2008; Yang, 2007). That is, their reasoning about fraction magnitude would, for the most part, fall under the Transform and Parts perspectives. Pilot interviews that we conducted with preservice elementary teachers who had completed their mathematics content courses confirmed this expectation. In our instructional sequence, we aimed for the more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation. In particular, we sought to engage students in reasoning about fraction size from Smith's Reference Point and Components perspectives. Tasks were designed and sequenced so as to begin with students' current ways of reasoning and to provide opportunities for reasoning about fraction magnitude in new ways. The design and sequencing of tasks involved consideration of these perspectives relative to number choices, contexts, and anticipated student reasoning, and guided by the envisioned learning routes described in our LIT. In the instructional sequence, we aim for these more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation.

Instruction

The course was intended to foster the development of number sense. In particular, the broad instructional intent was for students to come to act in a *mathematical environment* in which the properties of numbers and operations afforded a variety of computational strategies, as opposed to one in which mathematical operations map directly to particular algorithms (Greeno, 1991). Topics in the curriculum include quantitative reasoning around various story problems, place value, meanings for operations, children's thinking, standard and alternative algorithms, representations and values of rational numbers, and operations involving fractions. Our planning of instruction involved identifying in the curriculum opportunities to engage students in computational reasoning, as well as to facilitate rich discussions concerning students' strategies. For both mental computation and fraction comparison, shared sets of strategies were established via mathematical argumentation. These strategies were given agreed-upon names, and the class maintained a list of strategies with examples of each.

Preliminary Results

Our analysis of collective activity during the rational number unit is ongoing. At this point, we sketch the progression of fraction comparison strategies that were used by students and articulated in whole-class discussion over the first three days of the unit.

Amongst other tasks, students were asked to compare pairs of fractions, to order sets of more than two rational numbers (most of them represented as fractions), to reason about placement of fractions on a number line, to engage with others' reasoning about fraction magnitude, and to explicitly discuss and name their strategies for comparing fractions. On the second and third days of the unit, these activities often involved placing paper tags representing fractions on a string that represented a number line. The string was set up in the corner of the classroom and stretched from one wall to an adjacent wall. Tags were placed on the string to indicate the locations of those numbers on the number line.

Progression of Strategies

The "part-whole" meaning of a fraction was discussed in class on the first day of the rational number unit. Students were asked to discuss the meaning of a fraction, and the instructor recorded ideas on the board. A formal definition from the book was also presented. Students appealed to this part-whole meaning in many of their arguments concerning fraction magnitude. On the first day of the unit, two basic strategies for comparing fractions were used by students, discussed, justified, and named. These addressed the cases in which the fractions to be compared had either a common denominator or common numerator. Students named the strategy for the common denominator cases "Equal slices, more pieces." This name reflects a Parts perspective, wherein the common denominator is taken to indicate that same-sized "slices" of the whole are being considered, so that the greater numerator indicates the number of slices of that fixed size. Similarly, the strategy for reasoning in the common numerator cases was named "Equal number of pieces, different sizes."

Early instances of these basic strategies were described in terms of parts of a whole, often using contexts such as pies, cookies, or pizza. For example, Nancy described her reasoning to compare 3/4 and 3/7:

> Okay, well, if you take a pizza, and it's like the same size no matter how many pieces you cut it into. If you cut into four pieces, then they'd be bigger because you have less pieces. If you cut into seven, there's gonna be more, but each one is gonna be smaller. And, regardless of how many pieces it's cut into, you're only gonna eat three. Would you rather have three of the bigger ones or three of the smaller ones?

Arguments similar to Nancy's were articulated repeatedly in whole-class discussion. These arguments involved Parts reasoning and, in particular, were predicated on the idea that the number of pieces that a whole is partitioned into is inversely proportional to the size of those pieces. This idea was articulated repeatedly (in students' language) and functioned to justify "Equal number of pieces, different sizes" arguments.

By the second day of the unit, the class began pointing to the strategy names as a succinct way of describing students' reasoning. That is, the phrase, "We used 'Equal number of pieces, different sizes'" came to stand in for an argument such as Nancy's. There were also instances in which this strategy was used without being stated explicitly, especially when it was used in support of other, more elaborate arguments.

Building on their Parts comparisons, students repeatedly used the strategies of converting to a common denominator or common numerator. Whereas the procedure of converting to a common denominator was familiar to students from their previous mathematical education, converting to a common numerator was new for many students. Both of these strategies manifested in the discourse not as algorithms but as arguments that built on "Equal slices, more pieces" and "Equal number of pieces, different sizes," together with conversions to equivalent fractions, which were discussed and established early in the fraction unit. On the second day, these conversion strategies were named as "Make same number of pieces" and "Make same sized pieces." These names reflect a Parts interpretation of fractions, which tended to undergird students' arguments concerning fraction magnitude.

On the third day of the unit, students began to articulate Reference Point strategies. The first of these was the residual strategy: students compared proper fractions by comparing the residuals (distances from 1) of each. This strategy initially arose in the task of ordering 7/8, 9/10, and 8/9. Here, the residuals were unit fractions, and students compared these by established Parts reasoning. Students then extended the residual strategy to cases involving less trivial residuals, which sometimes required more sophisticated means of comparison. Other variations included comparing to benchmarks other than 1, such as 1/2 and 1/4; comparing distances above, rather than below, a benchmark; and *straddling* a benchmark, in which two fractions are compared by identifying one of them as less than the benchmark and the other as greater than it (Smith, 1995). The class eventually named this group of strategies "Difference Compared to Benchmark."

We offer an example of relatively advanced Reference Point reasoning from Day 3 of the fraction unit. Students were given the task of ordering 15/29, 4/9, and 8/15, and placing these on the number line. The class first established through argumentation that 4/9 was less than 1/2, while 15/29 and 8/15 were both greater than 1/2. They then had to find a way to compare 15/29 with 8/15. In whole-class discussion, Rebecca (R) and Julie (J) presented their argument that 8/15 was greater, while the instructor (I) facilitated the discussion:

> R: So, we think that 8/15 is larger than 15/29, uh, because we converted 8/15 – we multiplied the 8 and the 15 to get 16/30.
>
> J: Times two.
>
> R: Times two. Sorry.
>
> I: Does everybody see that step? Equivalence?
>
> R: And then compared that to 15/29. And we imagined a pie [circular spread gesturing] and shaded in the areas. And in both, the 16/30, there's 14 pieces shaded.
>
> J: Not shaded.
>
> R: Not shaded. And the same 14 pieces over there [referring to 15/29 as the instructor is notating this information on the whiteboard]. So, and then, more area is missing from – since the pieces are bigger in 14/29 than in 14/30.
>
> I: So, your conclusion then is that there's more missing here [pointing to 14/29 and the words "more missing" written on the whiteboard].
>
> R: Yeah.
>
> J: Yes.
>
> I: And how does that help you decide which of these is greater?
>
> J: Which means that 15/29 is not as great as – sorry, less than.
>
> I: Is less than 8/15?
>
> J: Yeah.

Rebecca and Julie's argument employs a circular area model for fractions ("a pie") under the Parts perspective. The students made use of the residual strategy. This instance of the strategy is particularly elaborate since the residuals are large, and a transformation is required in order for the residuals to be compared in the manner of "Equal number of pieces, different sizes."

Figure 1. Rebecca argues that 8/15 is larger than 15/29

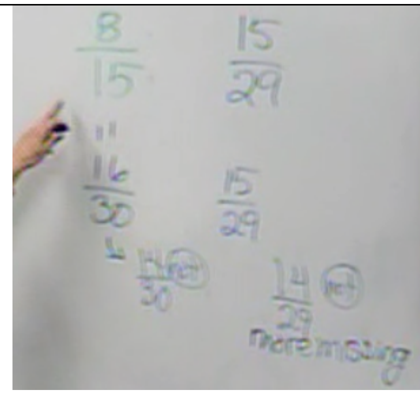

Figure 2. The instructor notates Rebecca and Julie's argument

Discussion

We present the example of Rebecca and Julie's argument for a few reasons:

1. The strategy that the students articulated is sophisticated and nonstandard.
2. The strategy is presented in the form of a complete, valid mathematical argument.
3. The students' argument builds on ideas that have been previously established in whole-class discussion: The residual strategy had been used previously in simpler cases. Producing equivalent fractions had been discussed and justified previously. Also, common numerator comparisons had been used repeatedly and crystallized in the form of a named strategy. This kind of comparison was used as an ancillary strategy in this case, and this seems to suggest that it functioned "as if shared" (Rasmussen & Stephan, 2008) by this point, although further analysis is needed to establish this.
4. At the same time, this example conveys the tentative nature of students' reasoning. Rebecca has some difficulty putting a complete argument into words (e.g., she says "shaded" when she means *not* shaded). Her facial expressions and voice inflection also suggest that this explanation requires care. Julie uses the language "not as great" rather than *less than*. These characteristics of the discourse are to be expected of prospective elementary teachers who are operating at the frontier of their knowledge of fraction magnitude. That is, they appear to be in the process of learning something.

Conclusion

We began this study with the idea of extending a local instruction theory for the development of number sense to the domain of rational numbers. In that vein, we shifted our focus from whole-number mental computation to reasoning about fraction magnitude. Smith's framework informed our thinking about strategies and perspectives involved in comparing fractions. The research literature, together with our pilot interviews, enabled us to anticipate prospective elementary teachers' initial reasoning about fraction magnitude. Our pre-instruction interviews enabled us to refine our expectations further. We designed the instructional sequence on the basis of students' starting points, conceived in terms of Smith's framework as essentially the Transform and Parts perspectives, and with the goal of building toward the Reference Point and Components perspectives. Specific instructional activities were then crafted with this broad progression in mind.

We found that we could apply the LIT to the rational number domain. By this, we mean that we found the LIT useful for our instructional design purposes during the course. The envisioned learning routes, incorporating Smith's framework, seemed to provide a viable way of conceptualizing students' learning. Our preliminary findings are encouraging, and our ongoing analysis involves documenting the collective activity in the class, using the methodology of Rasmussen and Stephan (2008). This analysis will enable us to investigate the relationship between the LIT and the actual classroom activity that it informs. This, in turn, will allow us to further elaborate the LIT in our continuing design research efforts concerning number sense development.

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