DYNAMIC VISUALIZATION OF COMPLEX VARIABLES: THE CASE OF RICARDO

Hortensia Soto-Johnson
University of Northern Colorado
Hortensia.soto@unco.edu
Michael Oehrtman
University of Northern Colorado
Michael.oehrtman@unco.edu
Sarah Rozner
University of Northern Colorado
Sarah.rozner@unco.edu

Quantitative reasoning combined with gestures, visual representations, or mental images has been at the center of much research in the field of mathematics education. In this research we extend these studies to include the arithmetic of complex numbers and the analysis of complex valued functions. Our data consists of videotaped interviews with experts, including mathematicians, physicists, and graduate students. Microethnography and phenomenological methods were used to analyze and interpret the data. In this case study we synthesize how one mathematician, Ricardo, employs geometric representations, gestures, metaphor, diagrams, and models to describe his understanding of complex variables topics. Linear transformations were fundamental in his connections between analytical and geometrical representations. He routinely characterized complex linear functions as linear transformations on $\mathbf{R}^2$ *with added symmetries. These findings may serve as a foundation for creating teaching experiments to help students develop geometrical representations of the mathematics behind complex variables.*

Key words: Complex variables, Embodied cognition, Experts, Perceptuo-motor activity

Introduction

The study of learning about numbers and their arithmetic operations is one of the best-developed fields in mathematics education research. The literature goes beyond the four basic operations to include composing and decomposing of whole numbers (Kilpatrick, Swafford, & Findell, 2001), verbal number competencies (Baroody, Benson, & Lai, 2003), ordering and comparing (Brannon, 2002), modeling and visual representations of the operations of whole numbers (Sowder, 1992). The research is not limited to whole numbers; rational numbers are part of the extensive literature related to number sense (Steffe & Olive, 2010). The studies on rational numbers entail investigating students' ability to create word problems that require division or multiplication of two fractions as well as students' visual representations of multiplication and division of two fractions. Studies of quantitative reasoning have elaborated the role of forming a mental image of the measurable attributes in a situation and conceiving of the relevant operations and relationships among these quantities (Thompson, 1994; Moore, Carlson, & Oehrtman, 2009). A natural extension to these studies is to investigate similar characteristics for complex numbers. This research is an effort to extend these studies and to contribute to body of knowledge on quantitative reasoning.

A long-term goal of our research is to create a framework, based on empirical evidence that describes how one perceives big ideas from complex variables. In an effort to contribute to this goal, we conducted an exploratory study to investigate the nature of experts' geometric reasoning about complex variables. In this report we describe one mathematician's flexibility with multiple representations of complex numbers and their operations including complex-valued functions. For our participant, Ricardo, these representations went beyond symbolism to include diagrams, gestures, perceptuo-motor activity, and metaphors. We explore how geometric representations, gestures, verbiage, and symbolism supports Ricardo's reasoning and how he uses these modalities to convey his understanding. As part of our results we found that complex linear transformations were central to Ricardo's reasoning about the arithmetic of complex numbers and analysis of complex valued functions.

Lierature Review

In this section we describe the few pieces of literature that hypothesize about students' understanding of complex numbers and a handful of studies that begin to provide empirical evidence about students' perspective of complex number arithmetic. We also include a description on how historical perspectives may influence the teaching and learning of complex variables. We begin by summarizing Sfard's (1999) theoretical article where she argued for the need for students to become more flexible in moving between operational and structural conceptions of complex numbers. She encouraged viewing the operational and structural components of complex numbers as complementary pieces rather than as dichotomous components. Researchers and instructors could support this perspective by integrating the two representations, for example through geometric illustrations of the operations since "visualization, … makes abstract ideas more tangible, and encourages treating them almost as if they were material entities" (Sfard p. 6). In order to transition from an operational to a structural perspective of complex numbers, Sfard posed three stages that students must navigate in order to develop their understanding of complex numbers.

The first stage is *interiorization*, which occurs when a process is performed on a familiar object. For the case of complex numbers Sfard claimed students who are just becoming proficient in using square roots, would be at the interiorization stage. *Condensation* is the second stage and it occurs when the learner is able to view a process as a whole without the tedious details. For example, students may continue to view $5+2i$ as a shorthand for certain procedures, but they would still be able to use this symbol in multi-step algorithms. The third stage, *reification*, is achieved when the learner has the ability to view a novel entity as an object-like whole. Learners who are at this stage would recognize $5+2i$ as a legitimate object that is an element of a well-defined set. According to Sfard (1999) this stage occurs as an "instantaneous leap" much like an "aha moment."

Danenhower (2006) incorporated both Sfard's (1999) framework and APOS theory to examine undergraduates' willingness and ability to convert variations of the fraction $\frac{a+ib}{c+id}$ to either Cartesian or polar form. Variations of the fraction included taking the absolute value of the numerator, raising the factors in the numerator and denominator to a power, and expressing the denominator in terms of sine and cosine, as well as other combinations of these variations. Danenhower found his participants were not flexible in moving between different representations and they did not tend to consider geometric representations, which would have made some items much easier to convert. Although the undergraduates were able to work with the polar form, they did not favor this representation due to the required use of trigonometry. Students were easily able to work with and frequently chose to work with the Cartesian form even though it was not as efficient as polar form. In summary, Danenhower found his participants worked with the Cartesian form of complex numbers at the object level, but could only work at the process level when complex numbers were represented in polar form. Both Danenhower's work and Sfard's analysis is restricted to introductory-level concepts about complex numbers. We intend to elaborate the interplay between more advanced concepts of complex numbers and complex-valued functions.

Lakoff and Núñez (2000) also offered a framework for the conceptual development of complex numbers. Their framework entails a conceptual blend of the real number line, the Cartesian plane, and rotations combined with the use of metaphor for number and number operations. Similar to historical descriptions, they portrayed multiplication of a real number x by –1 as a rotation of 180° to obtain $-x$. Thus, multiplying a number by i is equivalent to rotating by 90° counterclockwise. The beauty of this description is that mathematically it works, but empirical evidence suggests students do not view multiplying a number by –1 as a rotation of 180°, rather they perceive it as a reflection (Conner, Rasmussen, Zandieh, & Smith, 2007). This might be explained by the fact that students are focused on the real number line rather than the Cartesian plane.

In a more recent study, Nemirovsky, Rasmussen, Sweeney, and Wawro (in press) described the results of a teaching experiment with prospective secondary teachers enrolled in a capstone course. The goal of the teaching experiment was to create an instructional sequence that allowed students to create and discover the conceptual meaning behind adding and multiplying complex numbers. In this phenomenological study, the researchers incorporated microethnography to portray students' body activities over short time periods (between two to three minutes). These depictions included language use, gaze, gestures, posture, facial expressions, tone of voice, etc. As a result of their study, the researchers found perceptuo-motor activity was central in:

1. conceptualizing addition and multiplication of complex numbers,
2. communicating geometric representations for adding and multiplying complex numbers, and
3. creating a learning environment that influenced the development of structural components behind adding and multiplying complex numbers.

This study provides a hypothesis about how students make sense of arithmetic operations of complex numbers based on empirical data. As such it may provide insight into how best to introduce complex numbers to students besides as a tool for solving $x^2 + 1 = 0$. This study is particularly relevant to our research because we intend to create similar teaching experiments for advanced topics based on what we learn from experts. Incorporating reconstructed historical pieces of the development of complex numbers may also engender structural understanding of this area of mathematics (Glas, 1998).

Historically, the introduction and initial development of complex numbers was purely algebraic to resolve the issue of finding the real solution to certain cubic equations. Even after the square root of negative numbers was introduced, mathematicians such as Cardan found such numbers to be *sophistic* because they could not attach a physical meaning to these numbers (Nahin, 1998). These mathematicians tended to ignore the conceptual difficulties of these numbers and proceeded to apply the procedures "mechanically" (Glas, 1998, p. 368). It was Wallis who first dedicated much of his career attempting to represent the square root of a negative number through geometric constructions. Although, Wallis made progress his work was not convincing to other mathematicians or himself. It was more than a hundred years later that Wessel introduced the interpretation of placing $i = \sqrt{-1}$ at a unit distance from the origin on an axis perpendicular to the real number line to form the complex plane in which multiplying by i geometrically corresponds to a rotation of 90º counterclockwise. This representation allowed mathematicians to begin to think about complex numbers as vectors, which in turn led to geometric representations of the arithmetic operations of complex numbers. These models prompted mathematicians to prove that extended theories of complex numbers (i.e., quarternions, Cauchy-Riemann equations) are consistent and preserve the structure of the complex number system. For example, Hamilton's discovery of the quarternions was a result of wondering what would rotate vectors in three-dimensional space. Such historical developments may provide insights into how "concepts and theories can be best brought to light" for students (Glas, 1998, p. 377). Our theoretical perspective, discussed below, is in line with the small body of literature on historical and individual conceptual development of complex variables as described above.

Theoretical Perspective

As mentioned above Sfard (1999) did not view the operational and structural components of complex numbers as dichotomous, rather she viewed them as complementary pieces. In the same vein, some researchers (Lakoff & Núñez, 2000; Noe, 2006) argued that the way in which we perceive a mathematical concept and the way in which we physically interact with this concept are intertwined rather than disjoint, a perspective referred to as embodied cognition. Lakoff and Núñez contend that

> many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas. These include such ordinary cognitive mechanisms as those used for the following ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on. (p. 28)

Lakoff and Núñez offered four conceptual mechanisms typically employed in advanced mathematical thinking. The mechanisms are image schemas, aspectual schemas, conceptual metaphor and conceptual blend. Image schemas serve as a link between language and reasoning as well as language and visual perception. The notion of vision can be extracted from physical and real objects or something that is imagined i.e. a mental image. Aspect schemas occur when concepts that we associate as an action or event are conceptualized through motor-control schemas. As a result one communicates, prepares, gestures, etc. with a similar motor activity. In other words "the same neural structure used in the control of complex motor schemas can also be used to reason about events and actions" (Lakoff & Núñez, p. 35). The use of conceptual metaphors have been researched for over three decades in the domain of cognitive science and it has been established that the use of metaphors is systematic and unconscious. It has also been established that our personal and everyday experiences shape the metaphors that we use to convey abstract concepts through more concrete domains. The fourth mechanism, conceptual blend, entails combining "two distinct cognitive structures with fixed correspondence between them" (Lakoff & Núñez, p. 48). Metaphorical blends refer to a cognitive blend where the correspondence incorporates a metaphor, and are frequently used in the teaching, learning and understanding of mathematics.

By taking the perspective of embodied cognition we were better able to investigate how our participants incorporated general cognitive mechanisms used in their everyday nonmathematical world to craft their mathematical understanding and to communicate mathematical ideas related to complex variables. In the following section we discuss our research methods and follow this with our results.

Methods

We used purposeful sampling methods (Patton, 2002) to select our participants, which entailed selecting participants who we knew, either from personal experiences or through recommendations, possessed and articulated a rich geometrical understanding of complex variables. Although we interviewed several mathematicians, a physicist, and graduate students, this report details Ricardo's responses. Ricardo has over 20 years experience teaching undergraduate and graduate courses in complex analysis and his research interests include applications of partial differential equations, complex analysis, and Fourier analysis. He is known for having an extraordinary talent for teaching through metaphor and relating analytic and geometric representations in his teaching. The following quote describes Ricardo's commitment to and enthusiasm for developing students' visualization in a third semester calculus course:

> Because 3D visualization is a skill that needs constant nurturing, I have always made special efforts to engage their [students'] geometrical imagination, using a combination of low-tech and high-tech tools: by constructing paper, string, and wooden models of the surfaces we study, by drawing detailed multi-colored chalk diagrams that expose the geometrical structure of the objects under investigation, and by designing computer-based interactive labs and demonstrations that help students view geometrical objects from multiple perspectives. Each semester I hear gasps of sheer delight when students see a mathematical formula blossom on their computer monitor into a beautifully curved geometrical surface that they can explore at any level of magnification by rotating and zooming.

Data were collected through 90-minute video-taped interviews conducted by Hortensia and Michael. The interview questions (see Appendix 1) were structured to elicit how the participants geometrically perceived arithmetic and analytic concepts related to complex variables. The arithmetic questions centered on geometric representations of adding, multiplying, dividing, and exponentiating complex numbers. The analytic questions focused on geometric interpretations of continuity, the Cauchy-Reimann equations, differentiation, and contour integration of complex valued functions. At times, we probed in order to stimulate geometrical and visual explanations that our participants might provide to a novice. The process for analyzing and interpreting the data included transcribing the interviews while depicting where the participant incorporated visuals such as gestures, diagrams, models, and metaphors. After this, the research team used microethnography and phenomonological methods to analyze the video. This entailed watching and analyzing segments of the video repeatedly in order to capture and describe Ricardo's gestures and to synthesize his geometric explanations. We selected the segments based on their relevance to our research questions. We attempted to capture Ricardo's use of geometric representations throughout the entire interview unlike Nemirovsky et al. (in press) who only used small segments of the data. Upon completion of analyzing and interpreting the video we met with Ricardo in order to member check and to clarify any remaining questions. For example, at one point in time Ricardo made a movement with his hand for which we hypothesized several interpretations, so we asked him to explain what he was visualizing while he made the gesture. In the following section we summarize Ricardo's responses.

Results

In order to convey Ricardo's remarks we begin with a summary of his multiple representations for complex numbers and operations of complex numbers and complex valued functions. We follow this with a discussion of how these concepts and representations were embedded in his discussions related to differentiation of complex valued functions as well as his global perspective of complex analysis. It is important to note that we are not suggesting a hierarchical or dichotomous perspective of concepts. Rather, the data suggest that for Ricardo these representations and concepts were interwoven and complex linear transformations were the thread that allowed him to braid his perspectives.

Multiple Representations of Complex Numbers: Ricardo's flexibility to move from one representation of complex numbers to another was apparent in that he spontaneously recognized that some representations were more useful for given situations. During his interview, Ricardo represented complex numbers using vector notation, Cartesian and polar form, but he also made use of ordered pairs and points on the Argand plane. First and foremost he viewed complex numbers as vectors with the added property of multiplication and he commented that this was how he introduced complex numbers to students. Ricardo indicated that he only used the Cartesian form of complex numbers to add and subtract complex numbers and connected this to the geometric interpretation of vector addition and subtraction using a parallelogram. He also used his index finger or whiteboard marker to represent a vector and rotated his finger or the marker to indicate the rotation of a vector when multiplying by *i* (see *Figure* 1). For multiplication and division of complex numbers, Ricardo remarked that he preferred polar form because this allowed him to "visualize each of the pieces [the radius and the angle of the complex number] in my head." In essence it allowed him to recognize the "rotation and magnification factor." This was especially evident when we asked him to interpret the graduate students' processes for division of complex numbers. He had to pause and think about the geometry behind the graduate students' work before responding to Hortensia's question.

> Hortensia: Some graduate students attempted to locate 1z by rewriting it as zz2, what do you visualize when you see this notation?
>
> Ricardo: It's probably pure psychology, but I don't really process that one as quickly as the other one – the polar description, but the geometric description would be to first reflect across the real axis and then shrink it down by a scaling factor.

Complex variables textbooks rarely write complex numbers as ordered pairs, but Ricardo found this notation helpful in cases where he also represented complex linear transformations with matrices, especially in his discussion of the Cauchy-Riemann equations. Ricardo also thought of complex numbers as points representing a location on the Argand plane. He used this representation when he considered mappings and their input and output values in the plane. These representations are discussed further when we share his multiple illustrations of operations involving complex numbers and complex valued functions and when we portray his global view of complex analysis.

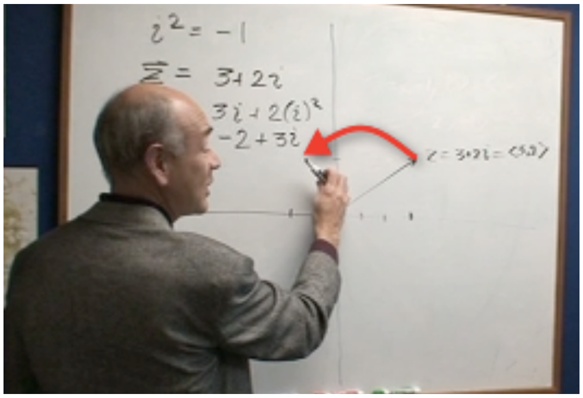

Figure 1. Using a marker to represent the rotation of a vector.

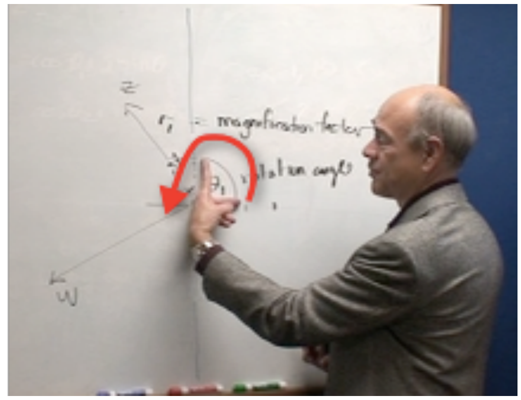

Figure 2. Rotating the first quadrant.

Multiple Representations of Operations: Two concepts seemed to be at the heart of Ricardo's ability to visualize and to bring to life complex variables concepts. The first was his recognition that multiplication by a complex number signified a rotation and dilation and the second was his conceptual understanding of functions. Ricardo persistently gave geometric interpretations and made corresponding gestures exemplifying multiplication by a complex number throughout the interview. Ricardo explained that he taught multiplication of complex numbers by first explaining how *i* acted on a specific complex number such as 3+2□ as depicted in *Figure* 1. After this he explained how multiplication acted on a frame, and consequently the entire Argand plane. In *Figures* 2 and 3, Ricardo configured his middle and index fingers as perpendicular vectors to represent a frame for the Argand plane and rotated his fingers to illustrate how multiplication by *i* would simultaneously transform both. He re-emphasized this representation when he discussed the geometric representation of multiplication of two general complex numbers *z* and *w*. After he described how multiplication acts on a frame, Ricardo progressed to illustrating rotating the entire plane. He used gestures similar to those shown in *Figures* 2 and 3 and swept his fingers around the entire plane. Another gesture that he seemed to favor to indicate a rotation of the plane was a fanned hand, which he rotated around the Argand plane as shown in *Figure* 4. He used this motion when he discussed the Cauchy-Riemann equations and when he discussed the notion of differentiation of complex valued functions.

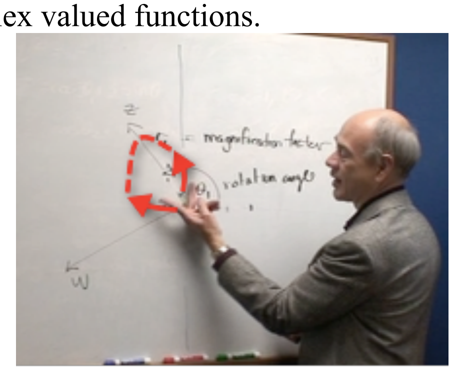

Figure 3. Rotating a frame to the second quadrant.

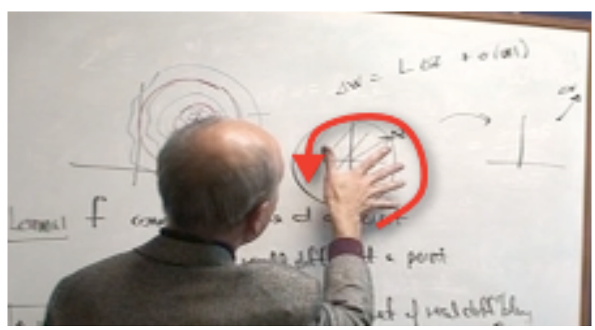

Figure 4. Rotating the Argand plane.

Ricardo also used hand gestures to represent expansions. There were two gestures that were prominent in his interview. The first began with a closed fist, which opened up to extended fingers as illustrated in *Figure* 5. The second gesture started with his hands together (see *Figure* 6) and then he pulled his hands away from each other. He repeatedly made these rotation and expansion motions when discussing multiplication. It was clear from his remarks and gestures that he perceived *z* as an operator and *w* as an operand in the product of *zw*, where *z* was the function that rotated and expanded the element *w*.

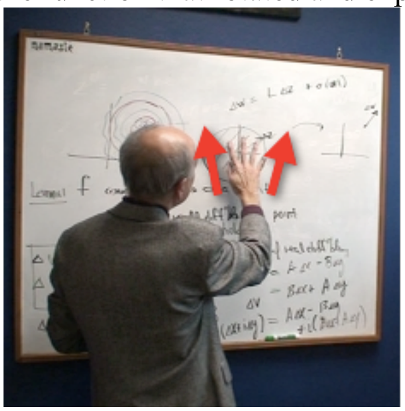

Figure 5. Expansion starting with fist.

Figure 6. Expansion starting with hands together.

The function concept was the second component that appeared to form Ricardo's geometric interpretations of the operation on complex numbers and concepts of complex variables. For example when we asked him about the location of □□, he incorporated the compositions of functions by first considering mapping of the points *z*, followed by rotations and dilations, and finally illustrating how exponentiation results in a spiral (see *Figure* 7). The following is his commentary that corresponds to *Figure* 7.

> If □=□□□□, then □□□=□□□+□(□+2□□) so if I plot all these points, all the values of log of *z*, they are separated by 2π in the imaginary direction. So it's a whole arithmetic progression in the complex plane that we're looking at. So when we multiply by *w*, in general if *w* is a complex number then it is going to be a rotation and expansion of this pattern here (points to 1st diagram). So when I look at *wlnz*, then possible interpretations of that are other arithmetic progressions on the plane and the spacing can be different from before because we performed an expansion by the magnitude of *w*. Now we want to exponentiate that (points to 2nd diagram) and that's actually going to be a logarithmic spiral (creates 3rd diagram).

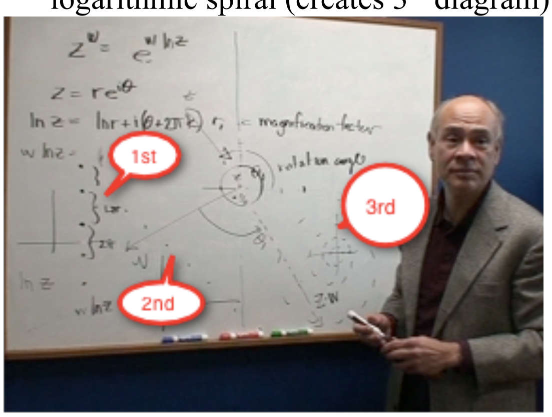

Figure 7. Illustrating □□.

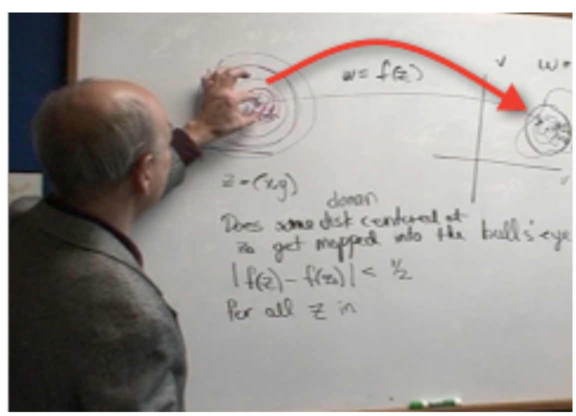

Figure 8. Demonstrating continuity by moving a chunk of space

Ricardo also demonstrated flexibility with multiple representations of operations when he responded to the continuity, Cauchy-Riemann equations and differentiation questions. In his geometric interpretation of continuity of complex valued functions he described how points that are close together in the domain get mapped to points that are close together in the range. Ricardo elaborated using the metaphor of a bulls' eye target centered around □□*0*=□*0*, where one would need to determine the radius of a disk centered at □*0* so that all the points in the disk get mapped inside the bulls' eye zone. As part of this metaphor he used his right index finger to illustrate mapping a point from the disk to the bulls' eye zone. This finger flicking movement illustrated the function acting on the domain point-by-point, but he also illustrated how it could map "*chunks of space*" at a time, by forming a c-shape with his left index and thumb moving it over to the range (see *Figure* 8). Ricardo continued to incorporate gestures when he discussed the continuity of the function □□=□□□□ at $z = 0$.

After reducing the function to □□=□□□□, he explained how one could graph this on a 3-D coordinate system, but before graphing it, he illustrated the output through gestures. The gesture appeared to be an infinity sign as shown in Figure 9, as if tracing the real-valued graph of $z = \cos 2\theta$ above the unit circle. After gesturing with this motion he moved to a physical model, which was the graph of two periods of the cosine wave going around a cylinder (in this case a thermos). He then proceeded to draw the vectors with different heights on the whiteboard. During the follow-up interview, he provided another model using paper, and explicitly confirmed the correspondence between his infinity-sign gestures and the thermos model. These models are shown in *Figures* 10 and 11. The arrow in *Figure* 10 outlines part of the curve he drew. During the follow-up interview Ricardo provided another metaphor that he found useful for describing continuity; it involved a painter's palette and count-by-numbers painting.

Ricardo's explanation of the geometry behind the Cauchy-Riemann equations was another example where he provided rich and multiple representations of his understanding. His representations included, matrices, systems of equations, ordered pairs, mappings, metaphors, gestures, and symbolism involving Landau notation. He commented that the underlying geometry of the Cauchy-Riemann equations occurred in the system of equations:

Δ□=□□+□□
Δ□=□□+□□

where Δ□=Δ□+□Δ□ is of first-order. He stated how

> this system of equations can be described by multiplication of a complex number if and only if this 2x2 matrix [matrix constructed from system of equtions] has certain symmetries. If we are lucky then □□ is the derivative of some constant times □. These symmetries are what allow one to deduce the Cauchy-Riemann equations.

As Ricardo discussed the symmetries he placed his hands at the center of the system of equations and pulled his hands away from each other in a diagonal direction to indicate the symmetries as illustrated in *Figure* 12. After this Michael asked Ricardo to re-interpret this with a picture on the complex plane, which generated a discussion about linear approximations.

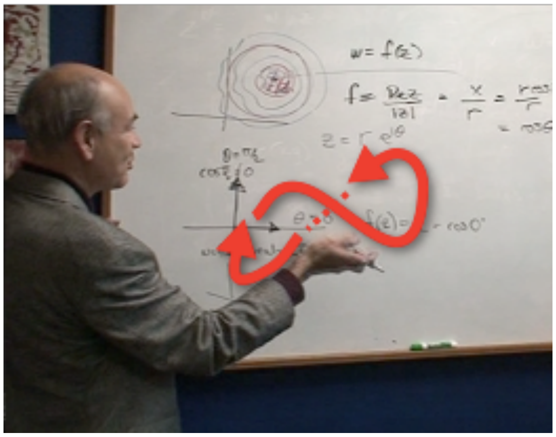

Figure 9. Discussing the continuity of □□=□□□□.

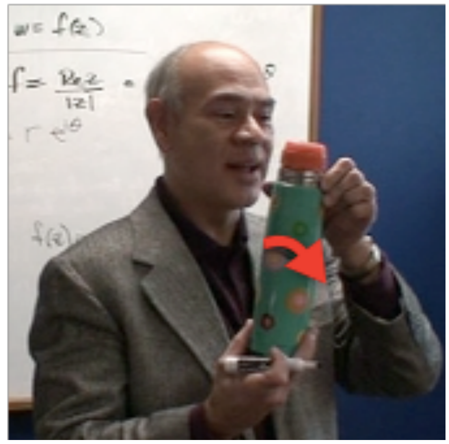

Figure 10. Physical model of □□=□□□□.

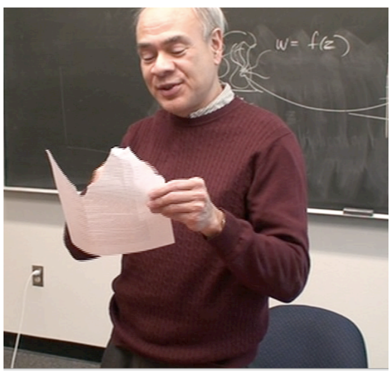

Figure 11. Physical model of □□=□□□□.

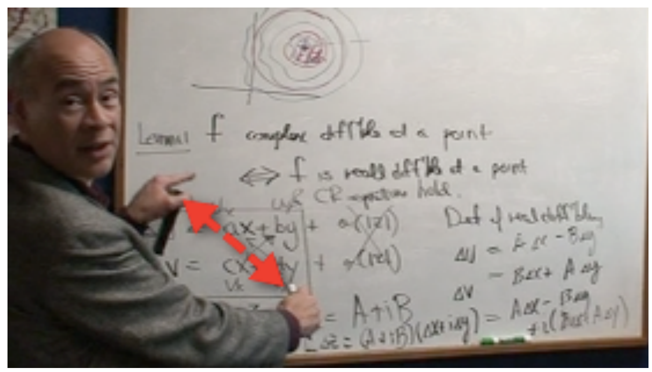

Figure 12. Depicting the symmetries of the C-R equations.

Ricardo discussed the first-order approximation Δ□=□Δ□ as a complex linear map , defined by a complex multiplier □=□+□□. Upon writing this he rotated his hand over Δ□=□Δ□, in the same manner as shown in *Figure* 4. He proceeded to explain how a disc with radius of □□ gets "rotated and expanded out" to produce the image in the range. After this description he used a turntable metaphor in which he said,

> it's like having a turntable that you can spin and expand and that is what the complex linear mapping is doing. So it's taking a patch of the plane and almost treating it like a rigid body rotating it, but then expanding it.

Figure 13, illustrates how he turned the turntable in a similar manner in which one might turn the steering wheel of a car. He followed this with his standard rotation and expansion gestures with spread fingers as described above. In this discussion, Ricardo commented that Δ□ is determined by the derivative as the product of some constant *L* and Δ□, which segued into the next interview question about differentiation.

As part of his geometric interpretation of complex differentiation, Ricardo reiterated that in general

> differentiation for any mapping means that a small patch can be approximated by what geometrically corresponds to a rotation and expansion. So small displacement vectors here (in the domain) are going to get rotated as a rigid body and then expanded uniformly.

As before, he explained this while using his rotation and expansion gestures. For the specific example of □□=□*2* and □′□= *2*□, Ricardo explored mappings of different disks centered at given point □*0* on the unit circle. He indicated that □*0* would get mapped to a point □*0* that has twice as big an angle due to the 2 in the exponent and that the disk centered at □*0* would double in radius and "get spun around" due to the 2*z* in the derivative with |*z*|=1. He illustrated this by drawing a small disk around □*0*, and noted that small displacements Δ□ in the domain will result in corresponding displacements Δ□ in the range. Specifically, the vector Δ□ will be rotated by 2arg(*z*) from the vector Δ□. Ricardo went on to remark that as one rotates around the unit circle in the domain of *f*, the amount of rotation varies depending on the angle of the point □*0.* As he stated this, he traveled around the unit circle in the domain with his left hand. He then moved to the range and explained that the "dial [image vector of z0] would get spun around to get different circles." This is an example where he used his left index finger to represent the vector and he moved it in a counter-clockwise direction around the unit circle in the range (see *Figure* 14).

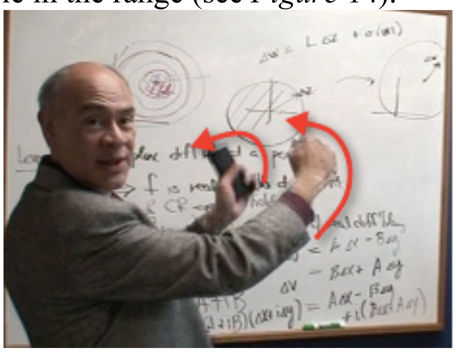

Figure 13. Turntable metaphor for complex linearization.

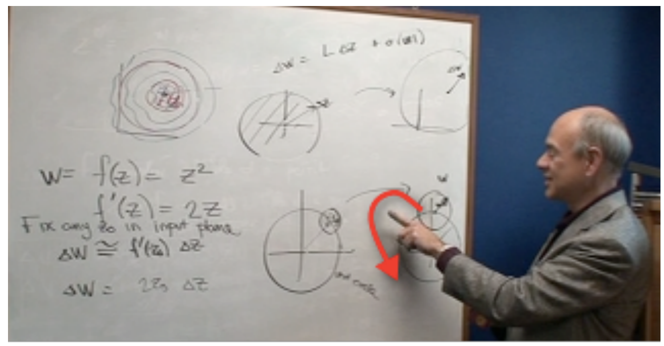

Figure 14. Spinner metaphor to describe differentiation at a point.

Real Analysis vs. Complex Analysis: We noticed that during the initial interview Ricardo always discussed and expressed rotation before dilation, with one exception. In the follow-up interview when we asked why this might be the case, the following dialogue occurred:

> Hortensia: Are expansion and rotation equally important or is one more important than the other for you?
>
> Ricardo: Theoretically they are equally important, but in my own psychology I am more interested by the rotation because it is more dramatic than the dilation (as he gestured as shown in *Figures* 4 and 5).

Although Ricardo initially called this a "personal quirk" he then emphasized how he perceived complex analysis as an extension of real analysis. This was prominent in his geometric interpretation of the product of two complex numbers, because in the real number system, multiplication only results in a dilation, whereas, multiplication of two complex numbers results in both a rotation and dilation. In the follow-up interview he emphasized this point,

> when you're analyzing complex differentiable maps if the derivative happens to be real then it's just a pure dilation (while he gestured as shown in Figure 5) and that is sort of the degenerate case and that's why I tend to look at the more general case that includes a rotation.

For Ricardo, differentiation in both systems represents an approximation for local linearity. He went on to say, "I think of it [complex linear maps] as a real linear map that has certain symmetries."

Figure 15, shown below, is an attempt to encapsulate Ricardo's perceptions of the arithmetic of complex numbers and the analysis of complex variables. Central to the model is linear transformations, since this appeared to be the foundation for Ricardo's geometric interpretation and visualization. Furthermore, it was the link between arithmetic and analysis and the link between real analysis and complex analysis. The model also exemplifies the significance that Ricardo placed on rotations and dilations – one can see the ample connections to and from this concept. Ricardo exemplified the four schemas described by Lakoff and Núñez (2000), and the dotted lines in the model depict how these schemas were not used in isolation. For example, Ricardo used the metaphor example of the Bulls' eye in conjunction with the thermos and paper cut-out concrete descriptions as part of his explanation about the continuity of complex-valued functions. We would like to re-iterate that Ricardo did not appear to possess a hierarchical view of these ideas; instead his representations appeared to be threaded together as one concept. This of course makes it even more difficult to portray Ricardo's perceptions through a model.

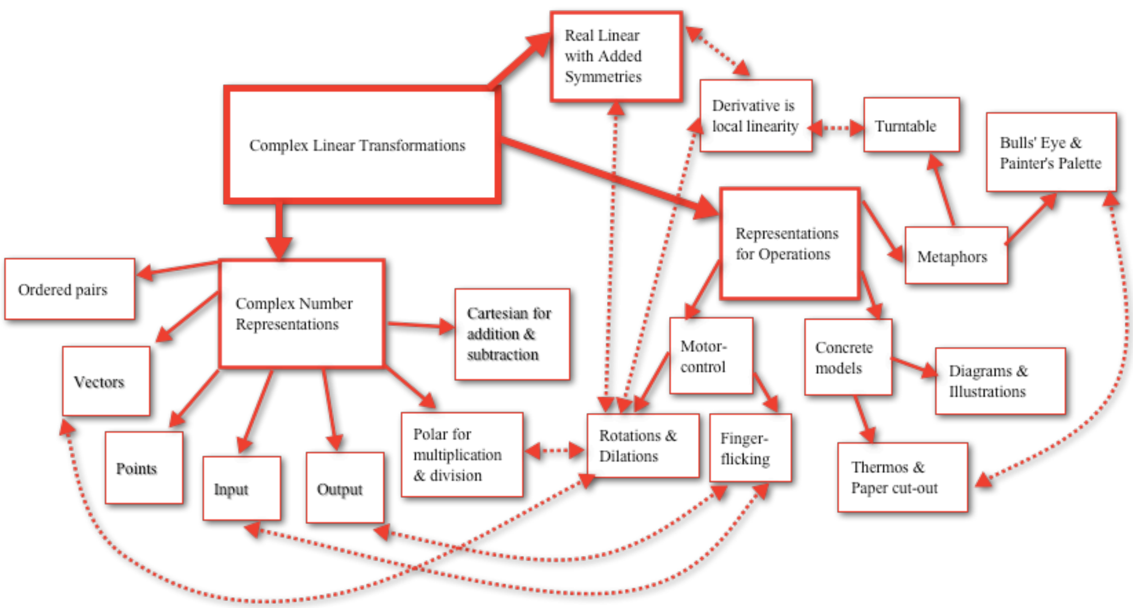

Figure 15. Illustration of Ricardo's geometric interpretation of complex variable.

Discussion and Future Directions

While Danenhower (2006) found undergraduates did not recognize the usefulness and appropriateness of different representations for complex numbers (i.e. Cartesian or polar), Ricardo easily navigated between different representations. He was conscious of the representations that he chose to implement in different situations; his purposeful and consistent choice appeared to be based on how the representation would aid Ricardo in visualizing the mathematics. The visualization and representation of the arithmetic and analysis of complex numbers and complex-valued functions went hand-in-hand. Sfard's (1999) work supports the non-dichotomous nature of Ricardo's understanding since both the visualization and representation of complex numbers or functions complemented one another. For example, given Ricardo visualized division of complex numbers as a rotation in the counterclockwise direction followed by scaling compelled him to represent *1*□ as *1*□□−□□. Similarly, even though he considered graphing □□ as "tricky," he used the fact that □□could be reinterpreted as a composition of multiple simpler transformations in order to reconstruct a visual diagram of the points represented by this expression. The ability to visualize each transformation allowed Ricardo to reify this novel mathematical situation.

Complex linear transformation played an unmistakable role in how Ricardo perceived complex variable topics and he consistently employed multiple conceptual mechanisms as described by Lakoff and Núñez (2000) to communicate his understanding. Ricardo incorporated image schemas that were concrete and abstract in nature. For example, his paper model and drawings and illustrations on the whiteboard and on the thermos were very concrete and easily visible. On the other hand his rotation and expansion gestures, his tracing along the diagonal to indicate the symmetries of the Cauchy-Riemann equations, his finger-flicking to illustrate mappings of points, and gestures for mapping regions on the plane were much more perceptuo-motor in nature and could be classified as aspect schemas since they were communicated through physical actions. Ricardo also conveyed his ideas through metaphors and conceptual blends that incorporated metaphors. These metaphorical blends appeared in his discussion of continuity through a bulls' eye target and a painters' palette as well as in his explanation of differentiation via a turntable and a spin-dial. Although we did not discuss Ricardo's explanation of complex integration, he also used a very elaborate metaphor. This metaphor referred to the path of a ship that traveled on a sea devoid of landmarks, a compass that deviated from true north, and a cable that expanded or contracted depending on water temperature. This description also relied heavily on his imagery of complex linear transformations discussed above. Although Ricardo's interview provided us with rich information and dynamic illustrations, it also triggered further questions.

One question that immediately comes to mind is are students attuned to Ricardo's conceptual mechanisms? As researchers, we only noticed some gestures after viewing the video numerous times. Thus, it is difficult to know if students actually connect the gestures to the mathematical content, especially if the gestures are different. As such, a follow-up research question is to investigate how, if at all, students make sense of others' embodied concepts that are expressed through conceptual mechanisms. A similar line of inquiry might be to investigate if and how experts, such as Ricardo, make a concerted effort to develop students' geometric interpretation of complex variables topics, especially those related to analysis. Our immediate plans are to complete the analysis of the interviews with the other mathematicians, physicists and graduate students and synthesize the results. We expect to obtain wide variations in the personal models such as Ricardo's illustrated in *Figure* 15. The next phase of our research will be to conduct guided reinvention teaching experiments with undergraduates to explore their geometric interpretations of the arithmetic and analysis of complex variables. Guided reinvention teaching experiments (*Gravemeijer* et al., 2000) are "a process by which students formalize their informal understanding and intuitions" (p. 237). These teaching experiments may allow us to capture a glimpse of students' embodied concepts of complex variable topics.

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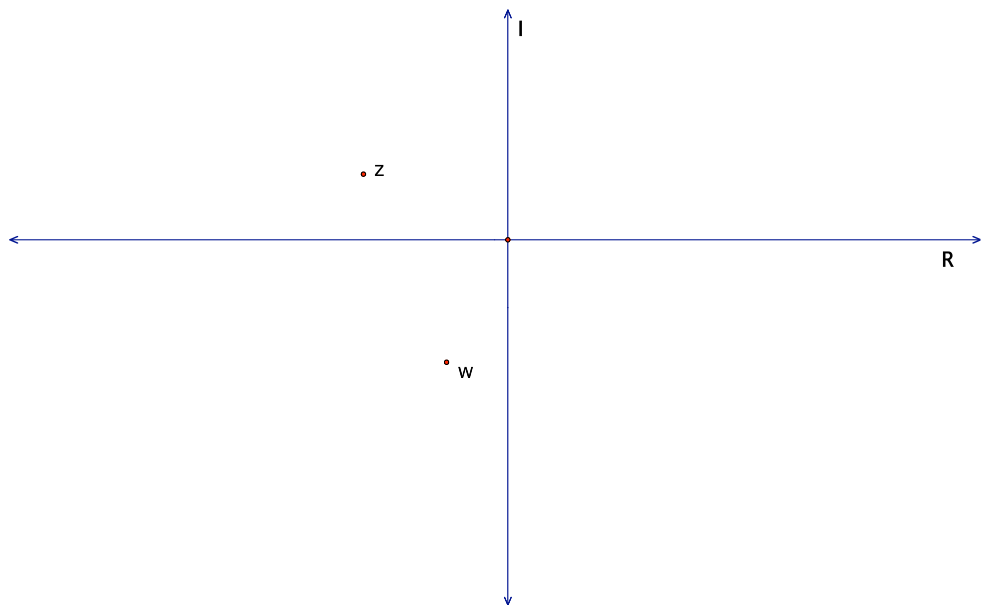
Appendix 1: Interview Questions

1. Below are two complex numbers z and w. Determine and explain how you know where each of the following are located. How do you think of these operations algebraically, geometrically, and in terms of $\mathrm{Re}^{i\theta}$?



   a. $z+w$
   b. zw
   c. $1/z$
   d. z^w

2. In calculus, we sometimes use the idea of "tracing the graph of function and not lifting our pencil" to convey the concept of continuity. What geometric representation or explanation might be useful to understand continuity of complex valued-functions?
   a. Consider the function $f(z) = \frac{\mathrm{Re}\, z}{|z|}$. Is there a way to define the function at $z = 0$

      in order to make the function continuous? Why or why not?

   b. Consider the function $f(z) = \frac{z \mathrm{Re}\, z}{|z|}$. Is there a way to define the function at z =

      0 in order to

      make the function continuous? Why or why not?

3. Give a geometric reasoning as to why it is enough for a real-differentiable

function $f(z) = u(x,y) + iv(x,y)$ to satisfy the Cauchy-Riemann equations in order for it to be complex differentiable throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$? Recall that the Cauchy-Riemann equations are: $u_x = v_y$ & $u_y = -v_x$

4. If $f(z) = z^2$, then
   a. What does it mean that $f'(z) = 2z$?
   b. What does it mean that $f'(2) = 4$?
   c. What does it mean that $f'(i) = 2i$?
   d. What does it mean that $f''(z) = 2$?
   e. What does the derivative of a complex function represent? Is it the slope of a line?
5. Sometimes in calculus we can interpret the definite integral of a real-valued function to represent the area under the curve. What geometric representation or explanation might be useful to understand the complex number obtained as an answer to a definite integral of a complex valued-functions?
6. How is complex variables useful in the real world? What are some applications of complex variables? How does it fit into the scheme of mathematics?