# EFFECTIVE STRATEGIES THAT UNDERGRADUATES USE TO READ AND COMREHEND PROOFS

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*In this paper, we examine the strategies that two successful undergraduate students used to comprehend six mathematical proofs. These students spent nearly as much time studying the theorem as they did studying the proof, both by reformulating the theorem and trying to understand why it was true prior to reading the proof. When reading the proof, these participants attended to proof frameworks, partitioned the proof, and considered specific examples. How these ideas might be used to improve students' proof comprehension is discussed.*

*Key words:* Proof; Proof comprehension; Undergraduate mathematics education

## 1. Introduction

In mathematical practice, proof plays both a role in the generation and communication of mathematical knowledge. For the individual mathematician, proof can be used as a tool, often an exploratory one, for coming to understand a piece of mathematics in several ways; proof can be used to gain conviction that a theorem is true, develop insight into why mathematical relationships hold, and discover new methods for attacking mathematical problems (see, for instance, de Villiers, 1990; Rav, 1999; Schoenfeld, 1994). For a community of mathematicians, proof is the predominant means of *communicating* mathematics. Most scholarly presentations and articles in professional mathematics are largely comprised of theorems and their proofs.

Proof is expected to play similar roles in mathematics classrooms, particularly in students' advanced mathematics courses (i.e., upper-level proof-oriented university mathematics courses). For elementary and secondary mathematics classrooms, the NCTM (2000) argues that justifying and proving should play an important role in all students' mathematics education throughout the curriculum; the NCTM advocates that all students appreciate the need for proof and be able to construct deductive arguments by the time they complete 12th grade (p. 56). Proof assumes an even larger role in advanced mathematics courses, where students' ability to prove theorems about the concepts covered in the course is often the primary means of assessing students' performance.

Not only are students expected to construct proofs in their advanced mathematics courses, but proof is also the primary means by which mathematics is conveyed to students. Lectures in advanced mathematics courses usually are delivered in a definition-theorem-proof format (e.g., Dreyfus, 1991; Weber, 2004, 2010), where students spend considerable time observing the professor present proofs to them. Textbook presentations typically place a similar emphasis on proof (e.g., Raman, 2004). Mathematics professors list a variety of reasons for engaging in this practice, including providing students with an understanding of why important theorems are true, illustrating new proof techniques to students, exposing them to proofs that have cultural significance in the mathematics community, and increasing students' appreciation of proof in general (Weber, 2010).

While there has been a great deal of educational research on proof, Mejia-Ramos and Inglis (2009) note that most of this research has concerned students' construction of proofs, with several researchers noting there is comparatively little research on students' reading of proofs and arguing more research in this area is needed (Hazzan & Zazkis, 2003; Mamona-Downs, 2005; Mejia-Ramos & Inglis, 2009; Selden & Selden, 2003; Weber, 2008). Mejia-Ramos and Inglis (2009) further note that research studies on students' reading of proofs have generally focused on the ways that students' *evaluation* of proofs rather than their *comprehension* of proofs. That is, researchers have typically investigated what types of arguments students found convincing or judged to be valid proofs. However, there are few studies on students' understanding of the proofs that they read. As an important goal of presenting proofs to students is to increase their mathematical understanding, the lack of research on students' proof comprehension represents an important void in the literature. The goal of this paper is to address this void by delineating strategies that undergraduate students can use to improve their comprehension of the proofs that they read.

## 2. Research context

### 2. 1. Overarching goals of our research program

The study reported in this paper was conducted in support of a larger research program. The overarching goal of this research program is to design and assess instruction that will improve undergraduate mathematics majors' ability to comprehend mathematical proofs. To achieve this goal, our research team seeks to (a) document the difficulties that undergraduates have in understanding mathematical proofs (see Weber, 2009, in press, for the results of this research), (b) identifying strategies that successful undergraduates use to overcome these difficulties, and (c) designing instruction that can enable less successful students to effectively employ these strategies to increase their comprehension of proofs. The report in this paper deals with the second goal of this research program—identifying strategies that successful students use to overcome these difficulties. We address this issue by reporting a case study of how two successful students read six mathematical proofs.

### 2. 2. Theoretical perspective

To identify strategies that help students understand a proof, it is necessary to first clarify what it means to understand a proof. In this paper, we adopt the theoretical model of understanding by Mejia-Ramos et al (2010). In this model, proof can be understood at an elemental level that focuses on the meanings of, or connections between, individual statements in the proof. Here, understanding can be assessed in terms of (a) meaning of terms and statements, (b) logical statuses of statements within a proof framework, or (c) how new statements were justified from previous statements. Proofs can also be understood in global terms, which include (a) understanding a summary of the proof, (b) being able to transfer methods of the proof to prove other theorems, (c) breaking the proof into components, or (d) applying the ideas of the proof to a specific example. A rationale for why these types of questions are important, as well as a scheme for generating these types of questions, is provided in Mejia-Ramos et al (2010). For the sake of brevity, they will not be discussed in this paper.

### 2. 3. The students



This study took place at a large state university in the northeast United States. Two students, Kevin and Tim (both names are pseudonyms), participated in this study. The students were each paid $20 an hour in exchange for their participation. Kevin and Tim were both mathematics majors in their senior year at the time of the study; both were also concurrently enrolled in the mathematics teacher education program at that university and preparing to become high school mathematics teachers.

Kevin and Tim were specifically invited to participate in this study for several reasons. Both students displayed an interest in mathematics, as well as an eagerness to study it, in mathematics education courses taught by the first author of this paper. In these classes, both students were articulate, successful, and willing to share their reasoning. Further, Kevin and Tim each participated in (separate and independent) research studies in the past at the university where this study took place and provided interesting and useful data on how some students were able to learn mathematics. In summary, Kevin and Tim were not typical students, but specifically recruited because we felt it was likely they would articulate interesting and effective proof-reading strategies.

During their interviews, Kevin and Tim revealed that they had taken advanced mathematics courses together in the past and were sometimes members of the same study group. We were not aware of this at the beginning of the study, but do not think this threatens the validity of our findings.

## 2. 4. The materials

All materials can be found in the Appendix of this paper. The materials consisted of six proofs of six theorems. Throughout the remainder of the paper, we refer to the $n^{th}$ theorem that the participants saw as Theorem n and its associated proof as Proof n. We designed these proofs to have the following features: (a) the mathematical content of the proof relied on introductory calculus and basic number theory (so participants' difficulty with the proofs would presumably not be due to a lack of content knowledge), (b) the proofs were of interesting propositions whose veracity would not be immediately obvious to an undergraduate student, and (c) the proof employed an interesting and non-standard technique. For each proof, we generated a set of proof comprehension questions using the proof comprehension model put forth by Mejia-Ramos et al (2009).

## 2. 5. Procedure

Kevin and Tim met with the first author of the paper for two task-based interviews, each of which lasted approximately two hours. In the interviews, Kevin and Tim were presented with a proof and asked to "think aloud" as they read and studied the proof. They were asked to read the proof until they felt that they understood it and, after they read the proof, they would be asked a series of questions about it to test their comprehension. When Kevin and Tim felt they understood the proof, the interviewer took the proof from the students, told the students he would ask them questions about the proof, and informed them that after the questioning was over, they would have the opportunity to read the proof again and change any of their answers to these questions. In this study, Kevin and Tim answered nearly every question correctly without the proofs in front of them, so there was no reason for them to change their answers upon viewing the proof again. This finding confirms that Kevin and Tim were effective at reading the proofs for comprehension. The participants were first asked the open-ended questions that follow each proof. They were then asked to answer the multiple choice questions that followed each proof. This process was then repeated. Each two-hour interview consisted of Kevin and Tim reading and evaluating three proofs.

## 2. 6. Analysis

As noted in the introduction, there are few research articles on proof comprehension (Mejia-Ramos & Inglis, 2009) and we are not aware of any research on the strategies that students should use to read proofs for comprehension. Consequently we did not have any pre-existing categories in mind when analyzing this data and opted to use an open coding scheme in the style of Strauss and Corbin (1990).

In a first pass through the data, we independently noted each attempt that Kevin and Tim made to make sense of the theorem statement or the proof and provided a summary of the students' behavior. (Here, "attempt" was construed broadly to mean anything beyond a literal reading of the text). After these summaries were produced, the authors met to discuss their findings.

From here, it was noted that Kevin and Tim's proof-reading could be divided into four phases: (a) studying the theorem, (b) reading the proof, (c) re-reading and summarizing the proof, and (d) critically evaluating the proof. These phases are discussed in more detail in section 3.1. Within each phase, similar proof-reading attempts were grouped together to form categories of the proof-reading strategies that Kevin and Tim employed. These strategies are discussed in detail in section 3.2. After categories were named and defined, we again independently viewed the videotape, coding for each instance of the proof-reading strategies. We then compared notes and discussed disagreements until they resolved. Most disagreements were the result of oversight on one of our parts. After our coding, Kevin and Tim were again interviewed about whether the strategies we observed were commonly used and why they engaged in those proof-reading strategies.

# 3. Results

## 3. 1. Phases of proof reading

Kevin and Tim's proof-reading could be partitioned into four distinct phases:

(a) *Studying the theorem*. This phase consisted of Kevin and Tim's behavior prior to reading the proof and was comprised of actions such as attempting to understand the theorem or see why the theorem was true.

(b) *Reading the proof*. This phase occurred as Kevin and Tim made their first pass through the proof.

(c) *Re-reading the proof*. On some occasions, after Kevin and Tim read and understood each step of the proof, as well as the proof as a whole, they would return to the beginning of the proof to re-read it. In some instance, they would simply check each step again. In others, they would form a high-level summary of the proof.

(d) *Critically evaluating the proof*. For Proof 1 and Proof 5, Tim would note apparent inconsistencies that he observed in the proof—either between assertions within the proof or between the arguments employed in the proof and his prior knowledge. Kevin and Tim would then discuss how those inconsistencies could be resolved.

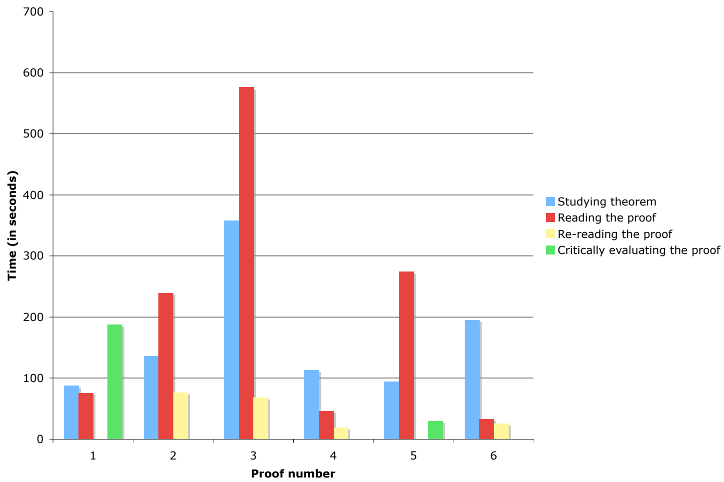


**Figure 1. Time spent on each activity for each proof**

The total amount of time spent on each phase of the activity for each proof is presented in Figure 1. We note two things about Figure 1. The first is the amount of time spent reading each proof. For each proof, Kevin and Tim spent between three minutes (proof 4) and over 16 minutes (proof 3) studying the proof, with an average of 7 minutes and 20 seconds per proof. This behavior seems to be atypical of how undergraduates studied proofs, as other studies report undergraduates generally spending under five minutes, and often less than two minutes, studying proofs (e.g., Selden & Selden, 2003; Weber, 2010). It is not clear if Kevin and Tim behaved differently from the undergraduates in Selden and Selden's (2003) and Weber's (2010) studies because they were unusually thoughtful and deliberate or because of differences in the task design (e.g., reading the proofs in pairs rather than alone or being told there would be a comprehension test after the proofs were read).

The second thing we note is the amount of time Kevin and Tim spent studying the theorem prior to reading the proof. Kevin and Tim averaged nearly three minutes (164 seconds) studying the theorem prior to reading the proof, spending nearly six minutes studying the statement of proof 3. While the participants spent a plurality of their time reading the proof itself (47%), it appears that the activity of studying the theorem was important as well. This suggests that proof-reading strategies should not only focus on how students read the proof of a theorem but also on how they comprehend the theorem.

## 3. 2. Strategies for proof-reading

In this section, we present the strategies that Kevin and Tim used in proof-reading. A summary of the students' strategy usage by proof is presented in Table 1.

**Table 1. Kevin and Tim's proof-reading strategies**

| Proof-reading Strategy | Proof 1 | Proof 2 | Proof 3 | Proof 4 | Proof 5 | Proof 6 |
|---|---|---|---|---|---|---|
| *Studying the theorem* | | | | | | |
| Rephrasing the theorem | | X | X | X | X | |
| Attempting to prove theorem | X | X | X | X | X | X |
| | | | | | | |
| *Reading the proof* | | | | | | |
| Identifying proof | | X | | | X | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| framework | | | | | | | | |
| Partitioning the proof | | | | X | | | X | X |
| Checking inferences with examples | | | X | | | X | | |
| *Re-reading proof* | | | | | | | | |
| Provide high-level summary | | X | | | X | | | |
| *Critically examines proof* | X | | | | | X | | |

***3. 2. 1. Studying the theorem***

*3. 2. 1. 1. Understanding the theorem*. Kevin and Tim would not proceed to reading a proof until they felt they understood what the theorem was asserting. This was illustrated sharply when the participants read the theorem statement for Proof 3, which introduced the concept of *k*-tuply perfect numbers. When Kevin suggests they read the proof, Tim stops him and says they first have to figure out what the theorem was saying. The theorem statement was:

We say that $n$ is $k$-tuply perfect if and only if $\sigma(n)= \sum_{d|n} d = kn$.

**Theorem:** If $n$ is 3-tuply perfect and 3 does not divide $n$, then $3n$ is 4-tuply perfect.

After spending some time trying to figure out exactly what was being asserted, Kevin became frustrated and suggested reading the proof.

Kevin: Well we just have to agree with the proof

Tim: Yeah, but we have to figure out what it [Theorem 3] is saying, so hold on.

*3. 2. 1. 2. Reformulating the theorem*. A common strategy that Kevin and Tim used to understand a theorem was to reformulate it. In some cases, this involved rephrasing the formal theorem less formally in their own words. For instance, when reading Theorem 3 (stated above), Kevin expressed the theorem as follows.

Kevin: So if n is 3-tuply perfect, so that's if all the divisors including itself add up to 3n, and if that's the case, and 3, just the number 3, does not divide whatever n is, then 3n, then 3 times it is 4-tuply perfect.

In other instances, Kevin and Tim would express what it would mean for the theorem to be true symbolically. For instance, Theorem 4 asserts "There is a real number whose fourth power is exactly one larger than itself". After reading the theorem, Kevin represented the theorem algebraically.

Kevin: So we're comparing x to $x^4$, right?

Tim: OK.

Kevin: And the $x^4$ power is exactly one larger than itself, right? So x + 1 is equal to $x^4$?

Tim: Is that right? [pause] Yeah.

Kevin: So this [$x^4 - x - 1$] would equal zero. So if we subtract x and 1 from both sides. So we know that $x^4 = x + 1$ based on the theorem…

Tim: We want to show that.

Kevin: We want to show that, right. So it should be based on something like this, and we would have to solve this.

In this instance, the reformulation that Kevin proposed suggested an approach for constructing the proof—namely, Theorem 4 was equivalent to showing that $x^4 = x+1$ had a solution. In fact, this was the approach used in Proof 4.

*3. 2. 1. 3. Seeing why the theorem is true*. Prior to reading the proofs of the theorems, Kevin and Tim would always attempt to prove the theorems themselves. For instance, consider how Kevin and Tim reasoned about Theorem 2, which asserted that if (a, b, c) is a Pythagorean triple, then c must be odd.

Tim: Before we do the proof, we're going to use some, what do you call it? Mod 2?

Kevin: Modular arithmetic?

Tim: I think so. Maybe.

Kevin: Why do you say we should be, we will be using…

Tim: Well you can also use 2k+1 and 2k and wind up being plus one at the end.

Kevin: Yeah.

Tim: OK. If a and b are both even and if you square them, those are both even and you add them, that's also even so c squared would become even. And they're not coprime.

Kevin: Exactly.

Tim: And the square root of an even number is…

Kevin: Right. If a and b are both even, c has to be odd or they're not coprime. So we can consider the other cases where either a and b are both odd or one is odd and the other is even. I mean, I assume that's what the proof is going to be. Those three cases, right? They're both even so c has to be odd. That one's easy..

Tim: Well odd squared plus odd squared is odd [sic] and two odds make an even.

Kevin: 2k + 1 squared plus 2l + 1 squared

Tim: And they can't be both odd… [pause] *That's interesting.* Anyway, anyway…

Kevin: We should probably just read. [italics are our emphasis]

In this excerpt, Tim suggests using parity as an approach to seeing why Theorem 2 is true. Parity-based arguments can be easily used to show why the theorem is true in all cases except when a and b are both odd. When seeing how this line of argumentation could not be used to address this case, Tim remarks, "That's interesting". Realizing why the obvious approach to proving Theorem 2 was insufficient appeared to motivate Tim and Kevin's reading of the proof.

Tim and Kevin's work may have had an additional benefit as well. When reading case 3 of the proof, Kevin remarked, "If a is odd, and then they go through the whole rigamorole, oh and they just said what I said. I agree with this". Having produced a similar argument himself, Kevin was able to quickly skim and comprehend the argument in the proof.

A second illustration of Tim and Kevin attempting to prove a theorem occurred when they read Theorem 1, which asserts that "$4x^3 - x^4 + 2\sin x = 30$ has no solutions".

Tim: As x gets really big, it gets dominated by the negative x to the fourth term. And it's a parabola going down basically and it's going to get modulated a little bit.

Kevin: Right. And sine of x is…

Tim: Periodic.

Kevin: "It is periodic so that wouldn't really affect it too much… out of the two functions, f(x) is the trumping one.

Tim: So in the long run, it's going…

Kevin: It's really f(x) that matters.

Tim: And the question is, does it reach 30.

Like their attempt to prove Theorem 2 presented above, a complete proof was not reached. However, in this case, Kevin and Tim recognized the high-level ideas of the proof—since $2\sin x$ is periodic (actually, more importantly, but not stated, $2\sin x$ is bounded), the key to the proof is showing that $4x^3 - x^4$ has a sufficiently low bound as well. The details in the proof itself, such as using differentiation to find the critical points of $f(x)=4x^3 - x^4$, can be seen as supporting these high-level goals.

Another benefit of attempting to prove the Theorems prior to reading the proof is that these efforts sometimes helped Kevin and Tim determine what proof frameworks (in the sense of Selden and Selden, 1995) would be appropriate to use in a proof. Consider Tim's proof attempt for Theorem 3:

Tim: So 3n is 4-tuply perfect because the sum of all the numbers that divide 3n, including 3n, sums to 12n, right?

Kevin: Say what?

Tim: Use the pen, the power of the pen… [Tim begins writing] such that d divides 3n is 4 times 3n… for n is 3-tuply. What do you think about that?

Tim: The numbers that divide n is 3n, right? Now if you take 3n, that's 4-tuply because sigma of 4 of 3n is the sum of the divisors of 3n.

Kevin: But we're proving that it's 4-tuply.

Tim: Right, right. This is true is what I'm trying to say. I'm just trying to say that if this is true, then this is true… I'm not sure if I'm dong the implications right.

Although Tim was unable to make much subsequent progress on constructing the proof, it did allow him and Kevin to recognize that they needed to show the factors of $3n$ summed to $12n$ using the fact that the sum of the factors of $n$ summed to $3n$, which was the method used in Proof 3.

Tim discussed several benefits of attempting to prove a theorem before reading its proof. He notes, "it makes you more of an active proofreader". Also, Tim noted that a practical reason for reading proofs is so he could expand his arsenal of proving techniques, noting one goal when reading a proof was "understanding the tricks and techniques so I can use them again". Mathematicians also list this as an important reason for reading proofs (Weber & Mejia-Ramos, 2011). Tim claimed that if he considered how an argument would proceed before reading it, it would help him "learn to do it [himself]"..

#### 3. 2. 1. 4. Discussion.

Before reading a proof, Kevin and Tim would sometimes reformulate the theorem being proven and would always attempt to prove the theorem themselves. Reformulation not only allowed Kevin and Tim to gain a meaningful sense of what the theorem was asserting, but also in some cases, suggested a method of proof. Dahlberg and Housman (1998) suggested that reformulations of definitions of concepts may help students construct proofs about that concept. It may also be the case that reformulating theorems can aid in the comprehension of proofs. Kevin and Tim consistently tried to prove the theorems themselves before reading the proof. This appeared to have a number of benefits, including motivating their proof-reading, allowing them to anticipate the higher-level structure of the proof, recognizing the proof frameworks that were employed in the proof, and appreciating the methods used in the proofs that they read.

### *3. 2. 2. Reading the proof*

#### 3. 2. 2. 1. Attending to proof frameworks

When reading the proofs, Kevin and Tim sometimes explicitly attended to what claim (or sub-claim) was being proven, what proof technique was being employed, and what the hypotheses and conclusions would have to be given that claim and proof technique. This was most clearly illustrated in their reading of Proof 5, where Tim was confused by the overarching structure of the proof. Kevin explicitly attends to each of the issues described above, as is illustrated both in their conversation and their written work on Proof 5 presented in Figure 2 below.

**Figure 2. Kevin and Tom's written work on Theorem 5 and Proof 5**

$P \longleftrightarrow Q$

The number of divisors of a positive integer $n$ is odd if and only if $n$ is a perfect square.
*Proof:*
1. Let $d$ be a divisor of $n$.
2. Then $\frac{n}{d}$ is also a divisor of $n$.
3. Suppose $n$ is not a perfect square.
4. Then $\frac{n}{d} \neq d$ for all divisors $d$, so we can pair up all divisors by pairing $d$ with $\frac{n}{d}$.
5. Thus, $n$ has an even number of divisors.
6. On the other hand, suppose $n$ is a perfect square.
7. Then $\frac{n}{d} = d$ for some divisor $d$.
8. In this case, when we pair divisors by pairing $d$ with $\frac{n}{d}$, $d$ will be left out, so $n$ has an odd number of divisors.

| | |
|---|---|
| $P$ | ~~$\mathbb{Z}^+$~~ Odd |
| $\neg P$ | Even |
| $Q$ | $k^2$ |
| $\neg Q$ | $\neg k^2$ |

Kevin: From [lines] two to four, it's doing the proof by contradiction. Suppose n is not a perfect square. […]

Tim: Suppose n is not a perfect square. So you're saying, suppose not this?

Kevin: For which one?

Tim: So you're saying, suppose n is not perfect, right, and that's the opposite of the right side.

Kevin: Cause it's dichotomous. If it's not a perfect square, then it's even. So therefore if P implies Q, then not Q implies not P, right? Contrapositive.

Tim: So we're saying this is P [writes P abover "the number of divisors of a positive integer n is odd], this is Q [writes Q over "n is a perfect square"].

Kevin: OK sure. So P is a positive integer, right? Positive integer?... Right?... Positive integer?

Tim: Is odd?

Kevin: Right. Sorry. P is odd, right? So then not P would be even. So Q would be a perfect square, k squared. And not Q would not be $k^2$ or whatever. So here, [referring to lines 3-5] it shows not Q implies not P.

Tim: And this part of the proof ends at 5 [Tim draws horizontal dashed between lines 2 and 3 and between lines 5 and 6 to partition the two parts of the proof]

Kevin: So it shows not Q implies not P and therefore P implies Q.

Tim: OK.

Kevin: So it's proving it forwards. [Tim rights a right arrow next to lines 3, 4, and 5]. It's the forwards way. [Reading line 6] On the other hand, suppose n is a perfect square. So now it's going to prove the backwards.

Selden and Selden (2003) and Weber (2009) found that students would often accept a proof of a conditional statement as valid in cases where the proof began by assuming the conclusion of the conditional statement and deduced the antecedent—i.e., the argument proved the converse of the statement. Both research teams took this as evidence that participants did not attend to the proof framework of the proofs that they were reading. The excerpt above illustrates that Kevin and Tim explicitly attend to the assumptions and conclusions of the proof to understand how a valid proof technique is being employed.

#### *3. 2. 2. 2. Partitioning the proof*

For proofs that could be broken into different modules, Kevin and Tim would sometimes partition the proof to see when one part of the proof began and the other part ended. For instance, the first eight lines of Proof 3 consisted of proving the lemma "if $m$ and $n$ are relatively prime, then $\sigma(m)\sigma(n) = \sigma(mn)$" and the last three lines of the proof were applying that lemma to Theorem 3. When reading the proof, Tim suggests partitioning the proof into parts and initially skipping the proof of the lemma.

> Kevin: Is this of the lemma?
> Tim: This is the proof of the lemma.
> Kevin: Where does the proof of the lemma end?
> Tim: Good question.
> Kevin: Is there more? [checks other side of the sheet]
> Interviewer: The proof of the lemma ends at step 8.
> Kevin: So why isn't this just the entire proof?
> Tim: Because when you use the lemma, you prove this.
> Kevin: So the actual proof is just like two steps. Well three steps. Well, nine to eleven, so it's four steps. OK. [Kevin begins reading the proof, starting at the lemma]
> Tim: So, let's not read the lemma first.
> Kevin: You have to read the lemma.
> Tim: Let's read the lemma later. Trust me.

In this instance, partitioning the proof into sections allowed Kevin and Tim were able to discern the high-level structure of the proof before reading its technical details. Their later comments revealed that Kevin and Tim both agreed that seeing how the lemma was used in the proof motivated their need to read the proof of the lemma. Kevin claimed a benefit of partitioning the proof into parts was that it helped break complicated proofs into chunks that were manageable for him.

#### 3. 2. 2. 3. Using examples to make sense of statements within the proof

For Proofs 3 and 5, when there was confusion about an inference within a proof, Kevin and Tim would attempt to resolve this confusion by seeing how the step applied to a specific example. For instance, the first two lines of Proof 5 were: "Let $d$ be a divisor of $n$. Then $n/d$ is also a divisor of $n$". Tim hesitated upon reading this statement, prompting Kevin to add:

> Kevin: So, say 7 is a divisor of 21. So, to get the other factor, it's 3.
> Tim: Right, right.

Kevin and Tim both emphasized the importance of using examples to make sense of problematic aspects of the proof with Kevin noting that while numbers cannot be used to prove a general statement, they are "really good to conceptualize it". In general, Kevin and Tim both emphasized that considering examples was essential when trying to make sense of abstract mathematics (particularly with proofs involving summations) because without using examples to make these ideas concrete, they would be unable to follow what was asserted within a proof. It is interesting to note that some mathematicians also claim to often work through a proof with a specific example (Weber & Mejia-Ramos, 2011).

#### 3. 2. 2. 4. Summary of proof-reading strategies

Kevin and Tim displayed three proof-reading strategies in their initial reading of the proofs—identifying the assumption and conclusions of the proof framework being used, partitioning the proof into parts, and using examples to understand problematic aspects of the proof. The first two strategies concern understanding the high-level structure of the proof—namely what are the main components of the proof and what is the logical structure of each of those components. The third concerns local details of the proof by trying to understand individual statements.

### *3. 2. 3. Summarizing the proof*

After reading each step of the proof, Kevin and Tim would sometimes return to the beginning of the proof and re-read it. In some cases, this process seemed merely to remind themselves of what the proof was saying and to check that each step was valid (possibly in preparation for the comprehension test they would be given shortly afterwards). In other cases, Kevin and Tim would give a summary that expressed the high-level ideas of the proof that went beyond describing what was in each individual step in the proof. After reading Proof 2, Kevin and Tim summarized it as follows:

> Tim: There are three possibilities
>
> Kevin: And two can't exist
>
> Tim: A and b are both even. A and b are both odd. A and b are even odd. The first two aren't possible, so it's got to be a even and b odd or b even and a odd.
>
> Kevin: And those are probably interchangeable.
>
> Tim: And the result of that is c is odd. And that's what we wanted to show.
>
> Kevin: Right. It has to be odd because it has the plus one.

Note that in this summary, the participants broke the proof into three separate cases and quickly dismissed two cases as impossible. Kevin ended the transcript by describing the method used to show the third case implied the desired result. When summarizing Proof 4 (which was only four lines long), Kevin strips away the specificity of the point to describe the overarching method used in the proof:

> Kevin: Right, so given that we know it's a continuous function, and some value makes it negative and some value makes it positive, so since it's continuous, it's got to go through zero at some point.
>
> Tim: Right.
>
> Kevin: And it doesn't actually say what it is, but that's irrelevant.
>
> Tim: [laughing] Right. All the [inaudible] wants to know is, "Does it exist?"
>
> Kevin: Right. Sounds good.

Note that Kevin observes an important detail of the proof. The root of the function cannot determined from the proof (i.e., the proof is not constructive) but both Kevin and Tim recognize that this is nonetheless sufficient to prove the theorem. In the excerpts highlighted above, we observe how the act of summarizing provided Kevin and Tim with insight about the higher-level structure and method involved in the proofs that they read.

### *3. 2. 4. Critically evaluating the proof*

After reading the proof, Tim would sometimes notice what he thought could be an inconsistency in the proof. In Proof 1, it is deduced that the function $f(x) = 4x^3 - x^4$ diverged to negative infinity as $x$ approached either positive infinity or negative infinity. It was also deduced that $f(x)$ contained exactly two critical points. Tim noted that if critical points were synonymous with turning points, this would be a contradiction.



> Tim: There are two critical points, so it's sort of like at the top and it sort of like

flattens out and keeps going down. Like that [draws a graph with three turning points].

Kevin: You mean like if it doesn't line up? You mean if it were like f(x) and g(x)?

Tim: No, no, a critical point is like where the derivative is equal to zero, which means its like horizontal slope. So I was just thinking it's like... I don't want to be too…

Kevin: [referring to Tim's graph] There would be three here. Here, here, and here.

Tim: [referring to the proof] But there's only two [critical points].

Kevin: Oh you mean a sort of saddle point?

Tim: Yeah, I forgot the word.

Kevin: Like $x^3$.

Tim: Right, there was only two critical points, so I was sort of like… [draws graph with one turning point, one saddle point, and diverging to infinity in both directions]

The resolution to the apparent contradiction was that a critical point is not necessarily a turning point. It could also be a saddle point. After reading Proof 5, Tim asks where the assumption in Theorem 5 that *n* is a positive integer was used in the proof. This might seem to be an inconsistency with Tim since proofs usually make use of all of their hypotheses.

Tim: Actually, real quick, can we talk about one more thing? Why does it matter that it's a positive integer?

Kevin: Well if it's a negative integer n, then you're dealing with negative factors.

In both instances, Tim pointed out what seemed to be an inconsistency that Kevin was able to help him resolve.

## 4. Discussion

### 4. 1. Caveats

This paper sought to identify effective proof reading strategies that could be taught to undergraduates in future teaching experiments. Of course, as with any case study, there are caveats due to the small sample size of this study. In particular, it is possible that using different proofs, or performing this study with different students, would yield different proof-reading strategies. Hence the proof-reading strategies described in this paper are not meant to constitute an exhaustive list, but can be used as a starting point for future teaching experiments.

### 4. 2. Viability of these strategies

There are (at least) three hypotheses for why the proof-reading strategies described in this paper might not be appropriate to teach to undergraduates:

(i) The strategies used in this paper are unique to Kevin and Tim and are not used by many other students who are effective at reading proofs.

(ii) Most undergraduates already use these proof-reading strategies. Hence, teaching undergraduates to use these strategies would not improve their proof-reading substantially. The keys to Kevin and Tim's success lie elsewhere.

(iii) Kevin and Tim's use of these strategies was helpful, but the implementation of these strategies relied on Kevin and Tim's conceptual understanding of the mathematics involved and/or with their experience in constructing proofs. Students without such a background might not benefit from using these strategies.

We discuss each hypothesis below. Regarding (i), in separate papers, we have examined the ways that professional mathematicians read proofs (Weber, 2008; Weber & Mejia-Ramos, 2011) and note consistencies between Kevin and Tim and the mathematicians' behavior. In particular, like Kevin and Tim, the mathematicians would also identify proof frameworks (Weber, 2008), partition proofs (Weber & Mejia-Ramos, 2011), and use examples to understand problematic aspects of the proofs they were reading (Weber, 2008; Weber & Mejia-Ramos, 2011). Further, there is some evidence that mathematicians would try to prove the theorem themselves to motivate the proofs that they read and summarize the proof by seeing how different sections of the proof related to one another (Weber & Mejia-Ramos, 2011). Hence some of the strategies that Kevin and Tim used where consistent with mathematicians who are presumably experts at comprehending proofs.

We also note that some of the strategies that Kevin and Tim used were consistent with suggestions from mathematics educators. For instance, Leron (1983) argued that proof comprehension would be facilitated by first presenting the high-level ideas of the proof and then supporting sub-arguments to carry out these high-level ideas. Kevin and Tim's partitioning of the proof could, in a sense, be viewed as an attempt to structure a proof themselves. Selden and Selden (2003) emphasized the importance of identifying proof frameworks when reading a proof and Kevin and Tom explicitly attended to this.

Finally, we note that Kevin and Tim's learning strategies seemed consistent with the proof comprehension model of Mejia-Ramos et al (2009). The strategies they used seemed to have direct benefits for understanding the meaning of a theorem, identifying proof structures, providing high-level summaries of the proof, describing the general method used in a proof, and applying the ideas of the proof to a specific example. It is possible that the assessment items that were based on Mejia-Ramos et al's model influenced the participants' proof-reading behavior. However, we note that some of the proof-reading strategies were used when reading Proof 1 (prior to receiving any assessment items) and the participants described the strategies that they used in this study as ones they used frequently.

Regarding hypothesis (ii), the studies of Selden and Selden (2003) and Weber (2010) suggest that undergraduates do not use the proof-reading strategies described in this paper. For instance, both Selden and Selden (2003) and Weber (2010) noted that many undergraduates ignore proof structure when reading a proof, preventing them from recognizing some proofs as invalid. Selden and Selden (2003) also observed that students tended to focus on local details of the proof (i.e., specific inferences and calculations) while ignoring higher-level details. In Weber's (2010) study, participants rarely considered examples or graphs when reading a proof, something that Kevin and Tim did in this study.

We do not have data to assess the viability of hypothesis (iii). Whether undergraduates possess the background knowledge to successfully implement Kevin and Tim's proof-reading strategies, or whether they can be taught this knowledge in a reasonable period of time, are questions that we will investigate in future teaching experiments.

## 4. 3. Teaching suggestions

From our perspective, one of the most surprising findings of our study is the importance that Kevin and Tim paid to understanding a theorem and attempting to generate a proof themselves before reading the proof in the text. In lectures, professors might consider spending some time reviewing the meaning of the theorem they are going to prove. Selden and Selden (1995) document that undergraduate students often are unable to extract the logical meaning of informal mathematics statements. It may be worthwhile for professors to guide students through this process. Professors may also want to give students the opportunity to understand why a theorem is true and generate their own proofs of the theorem before presenting a proof in the class. Kevin and Tim cited many benefits to doing this, including motivating the need for a proof and appreciating the higher-level ideas and methods used in the proof. Similarly, Leron and Dubinsky (1995) suggest that presenting a formal proof of an important theorem to a class after they have extended experience exploring that theorem as this might lead students to appreciate and understand the proof better.

Having students read and discuss proofs collaboratively might also benefit students. Throughout our study, we saw Kevin and Tim working together to overcome confusions that arose as they were reading the proof. Finally, as Conradie and Frith (1990) suggest, asking students deep and meaningful questions about the proofs that they read might not only encourage students to study these proofs more, but also inform them about what it means to understand a proof.

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## APPENDIX: Theorems and proofs used in this study

**Theorem 1.**

Claim: $4x^3 – x^4 + 2\sin x = 30$ has no solutions.

**Proof 1.**

Consider the functions $f(x) = 4x^3 – x^4$ and $g(x) = 2\sin x$..

Since *f(x)* is a polynomial of degree 4 whose leading coefficient is negative, $f(x)\rightarrow\infty$ as $x\rightarrow\infty$ and $x\rightarrow-\infty$. Hence, *f(x)* will have an absolute maximum. Taking the first derivative of f(x) yields $f'(x)=12x^2 – 4x^3$. Setting $f'(x)$ equal to zero and solving for $x$ yields $x = 0$ and $x = 3$. These are the critical points of *f(x)*. The absolute maximum must occur at a critical point. $f(0) = 0$. $f(3) = 27$. Hence, $f(x) \leq 27$ for all $x$.

The range of sin $x$ is [-1, 1]. Hence the range of 2sin $x$ is [-2, 2]. Therefore $g(x) = 2 \sin x \leq 2$ for all $x$.

$4x^3 – x^4 + 2\sin x = f(x) + g(x)\leq\max(f(x)) + \max(g(x)) = 27+2= 29<30$. Therefore $4x^3-x^4+2\sin x = 30$ has no solutions.

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**Theorem 2.**

Let a, b, and c be natural numbers. (a, b, c) is a Primitive Pythagorean triple if two conditions hold:

(1) $a^2 + b^2 = c^2$

(2) a, b, and c are co-prime (i.e., there is not a common factor of a, b, and c other than 1)

For example, (3, 4, 5) and (5, 12, 13) are Pythagorean triples. (9, 12, 15) is not a Pythagorean triple because 3 is a factor of 9, 12, and 15. (4, 5, 7) is not a Pythagorean triple because $4^2 + 5^2 = 41 \neq 49 = 7^2$.

Theorem: If (a, b, c) is a Primitive Pythagorean triple, then c is odd.

**Proof 2.**

Lemma 1: $a^2\equiv1 \pmod 4$ if a is odd and $a^2\equiv0 \pmod 4$ if a is even.

*Proof.* Suppose a is odd. Then a = 2k+1 for some integer k. $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So $a^2\equiv1 \pmod 4$ . Suppose a is even. Then a = 2k for some integer k. $a^2 = 4k^2$ so a is divisible by 4 and hence $a^2\equiv0 \pmod 4$.

Lemma 2: If $a^2$ is odd, then a is odd.

*Proof by contraposition.* It suffices to show that if a is not odd, then $a^2$ is not odd. This is equivalent to showing if a is even, then $a^2$ is even. Suppose a is even. Then $a^2$ is even since the product of two even numbers is even. This proves the claim.

*Proof of theorem* (by cases).

There are three cases. Either a and b are both even, a and b are both odd, or a is even and b is odd (or vice versa).

*Case* 1 (a and b are both even). Since a and b are even, $a^2$ and $b^2$ are even. Hence $a^2 + b^2$ is even. If (a, b, c) is a Pythagorean triple, $a^2 + b^2 = c^2$ so $c^2$ is even. By lemma 2, since $c^2$ is even, c must be even. Hence, a, b, and c are all even and hence all have a factor of 2. Thus (a, b, c) is not a primitive Pythagorean triple. There are no primitive Pythagorean triples if a and b are both even.

*Case* 2 (a and b are both odd). By lemma 1, since a and b are odd, $a^2\equiv1 \pmod 4$ and $b^2\equiv1 \pmod 4$. Thus, $a^2 + b^2\equiv2 \pmod 4$. If (a, b, c) is a Pythagorean triple, $a^2 + b^2=c^2\equiv2 \pmod 4$. By Lemma 1, we know that $c^2$ cannot be congruent to 2 (mod 4). Thus there are no primitive Pythagorean triples if a and b are both odd.

*Case* 3. (a is odd and b is even, or vice versa). Without loss of generality, assume a is odd and b is even. Then $a^2$ is odd and $b^2$ is even. Hence $a^2 + b^2$ is odd. If (a, b, c) is a Primitive Pythagorean triple, $a^2 + b^2 = c^2$ is odd. Since $c^2$ is odd, by Lemma 2, c is odd.

Hence, the only way to a Pythagorean triple can be formed is if a is odd and b is even or a is even and b is odd. In either case, c is necessarily odd.

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**Theorem 3.**

We say that $n$ is $k$-tuply perfect if and only if $\sigma(n)= \sum_{d|n} d = kn$.

If $n$ is 3-tuply perfect and 3 does not divide $n$, then $3n$ is 4-tuply perfect.

**Proof 3.**

1. **Lemma:** If $m$ and $n$ are relatively prime, then $\sigma(mn)=\sigma(m)\sigma(n)$.
2. *Proof:* let the prime factorization of $n$ be $n=p_1^{a1}p_2^{a2}...p_k^{ak}$. .
3. Then the divisors of $n$ are all numbers of the form $d=p_1^{b1}p_2^{b2}...p_k^{bk}$.where $0\leq b_i\leq a_i$.
4. But these numbers are precisely the terms in the expansion of the product: $(1+p_1+p_1^2+...p_1^{a1})(1+p_2+p_2^2+...p_2^{a2})\ldots(1+p_k+p_k^2+...p_k^{ak})$.
5. Thus, it must be that $\sigma(n)=(1+p_1+p_1^2+...p_1^{a1})(1+p_2+p_2^2+...p_2^{a2})\ldots(1+p_k+p_k^2+...p_k^{ak})$
6. Since each set of parentheses contains a geometric sum, we can rewrite this as $\sigma(n) = \frac{p_1^{a_1}-1}{p_1-1} \bullet \frac{p_2^{a_2}-1}{p_2-1} \cdots \frac{p_k^{a_k}-1}{p_k-1}$.
7. For $n$ and $m$ relatively prime, we will have $mn=p_1^{a1}p_2^{a2}\ldots p_k^{ak}q_1^{b1}q_2^{b2}\ldots q_j^{bj}$ where $n=p_1^{a1}p_2^{a2}\ldots p_k^{ak}$ and $m= q_1^{b1}q_2^{b2}\ldots q_j^{bj}$.
8. Then

$$\sigma(mn) = \frac{p_1^{a_1}-1}{p_1-1} \bullet \frac{p_2^{a_2}-1}{p_2-1} \cdots \frac{p_k^{a_k}-1}{p_k-1} \bullet \frac{q_1^{a_1}-1}{q_1-1} \bullet \frac{q_2^{a_2}-1}{q_2-1} \cdots \frac{q_j^{a_j}-1}{q_j-1} = \sigma(m)\sigma(n)$$

9. With this lemma in hand, suppose that $\sigma(n) = 3n$ and $n$ is not a multiple of 3.
10. Then $\sigma(3n)=\sigma(3)\sigma(n)$ .
11. However, $\sigma(3)=1+3=4$. So $\sigma(3n)=4\sigma(n)=4(3n)$,which makes $3n$ 4-tuply perfect.

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**Theorem 4.**

There is a real number whose fourth power is exactly one larger than itself.

**Proof 4.**

1. Let $f(x)=x^4-x=1$.

**2.** $f(1)=-1$ and $f(2)=13$.
**3.** By the intermediate value theorem, there must be some number $c$ such that $1<c<2$ and $f(c)=0$.
**4.** $c$ has the desired property.

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**Theorem 5.**
If every even integer greater than 2 can be expressed as the sum of two primes, then every integer greater than 5 can be expressed as the sum of three primes.

**Proof 5.**
1. Assume every even integer greater than 2 can be expressed as the sum of two primes.
2. Let $n>5$.
3. We proceed by cases.
4. Case 1: $n$ is even.
5. Then $n$-2 is even and greater than 2.
6. So $n-2=p_1+p_2$, with $p_1$ and $p_2$ prime.
7. Then $n = p_1+p_2+2$, a sum of three primes.
8. Case 2: $n$ is odd.
9. Then $n - 3$ is even and greater than 2.
10. So $n-3=p_1+p_2$, with $p_1$ and $p_2$ prime.
11. Then $n = p_1+p_2+3$, a sum of three primess.

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**Theorem 6.**
The number of divisors of a positive integer $n$ is odd if and only if $n$ is a perfect square.

**Proof 6.**
1. Let $d$ be a divisor of $n$.
2. Then $n/d$ is also a divisor of $n$.
3. Suppose $n$ is not a perfect square.
4. Then $n/d \neq d$ for all divisors $d$, so we can pair up all divisors by pairing $d$ with $n/d$.
5. Thus, $n$ has an even number of divisors.
6. On the other hand, suppose $n$ is a perfect square.
7. Then $n/d = d$ for some divisor $d$.
8. In this case, when we pair divisors by pairing $d$ with $n/d$, $d$ will be left out, so $n$ has an odd number of divisors.