# HOW MATHEMATICIANS USE DIAGRAMS TO CONSTRUCT PROOFS

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*Although some researchers argue that diagrams can aid undergraduates' proof constructions, most undergraduates have difficulty translating a visual argument to a formal one. The processes by which undergraduates construct proofs based on visual arguments are poorly understood. We investigate this issue by presenting eight mathematicians with a mathematical task that invites the construction of a diagram and examine how they used this diagram to produce a formal proof. The main findings from the paper were that it was not trivial for mathematicians to translate an intuitive argument into a formal proof and mathematicians used diagrams for multiple purposes, including noticing mathematical properties, verifying logical deductions, representing ideas or assertions, and suggesting proof approaches.*

*Key words*: Mathematicians, informal diagrams, proof construction

## 1. Introduction

### 1.1. Student difficulty with proof

Proving is central to mathematical practice. Hence, a primary goal of advanced undergraduate mathematics courses is to improve students' abilities to construct proofs. Yet numerous studies have documented undergraduates' frequent inability to construct mathematical proofs (e.g., Alcock & Weber, 2010a; Hart, 1994; Moore, 1994; Recio & Godino, 2001; Weber, 2001; Weber & Alcock, 2004). Research in this area has generally focused on particular difficulties that undergraduates face when writing proofs, such as having a poor understanding of advanced mathematical concepts (Hart, 1994; Moore, 1994), lack of proving strategies (Weber, 2001), and not knowing where to begin when asked to write a proof (Moore, 1994). However, research on how students can or should engage successfully in proof construction tasks has been sparse. In this paper, we examine in detail one specific suggestion for proof construction from the mathematics education literature—using a diagram as a basis for constructing a formal proof. We discuss the literature on this in the next section.

### 1.2 Diagrams in mathematics

Diagrams are viewed by mathematicians and mathematics educators alike as an integral component of doing and understanding mathematics (e.g., Hadamard, 1945; Stylianou, 2002). A substantial benefit that diagrams afford is they allow the problem solver access to view, compare, and integrate simultaneous pieces of information with little cognitive effort, something that is difficult when the same information is presented symbolically and sequentially (Eisenberg & Dreyfus, 1991; Larkin & Simon, 1987) – while making certain properties of mathematical concepts transparently obvious that would ordinarily be difficult to discern with non-visual representations of this concept (e.g., Piez & Voxman, 1996). Further, diagrams are often used to provide novel and more accessible explanations for mathematical phenomena or highlight aesthetics that are less accessible through symbols and logic (e.g., Hersh, 1993) as is illustrated by Nelsen's (1993) *Proofs Without Words*. Mathematicians' self-reports reveal that diagrams play a significant role in their mathematical work (e.g., Hadamard, 1945). Stylianou's (2002) investigation into mathematicians' problem solving revealed four purposes for diagrams: inferring consequences, elaborating upon mathematical ideas, creating sub-goals, and encouraging metacognitive reasoning.

Drawing diagrams is commonly cited as a heuristic for mathematical problem solving and reasoning that students should engage in (e.g., Polya, 1957; NCTM, 2000; Schoenfeld, 1985). However, despite the promise of this recommendation, the literature suggests that at least several pedagogical issues that must be addressed. First, researchers have noted that students are often reluctant to use diagrams, even for problems where their use might be highly productive (e.g., Eisenberg & Dreyfus, 1991; Piez & Voxman, 1996; Stylianou & Silver, 2004). Second, Presmeg (1986) found little correlation between high school students' propensity to visualize and their mathematical performance. Indeed, the strongest students in Presmeg's sample rarely used diagrams. Finally, Schoenfeld (1985) demonstrated that the process of implementing this type of heuristic is surprisingly complex; he argued that more explicit instruction on using diagrams (e.g., what to read from diagrams) might be necessary for students to employ this heuristic effectively.

**1.3. Diagrams and formal proofs**

Instructors of advanced undergraduate courses often ask their students to produce formal proofs. These proofs are expected to begin with definitions, axioms, and appropriate assumptions and proceed deductively to reach a desired conclusion, often while employing logical syntax. While informal representations of the involved concepts, including diagrams, may be included with the presentation of the proof, their role is expected to be ancillary, serving as a comprehension aid for the formal argument. The inferences within the proof are expected to be based on deductive logic, not the appearance of the diagram.

Both mathematicians and researchers in mathematics education emphasize that although formal proof results from proving, the process of proving can be, and frequently is, based on informal argumentation – often involving visual reasoning (e.g., Alcock, 2010; Burton, 2004; Boero, 2007; Dreyfus, 1991; Thurston, 1994; Raman, 2003; Schoenfeld, 1991; Weber & Alcock, 2004). Consequently, some researchers have suggested that undergraduates be encouraged to base the proofs that they construct on informal reasoning (Boero, 2007)—in particular by utilizing diagrammatic reasoning (Alcock 2010; Gibson, 1998; Raman, 2003; Weber & Alcock, 2004).

The literature supports this suggestion theoretically and empirically. Theoretically, researchers have noted benefits offered by diagrammatic reasoning in formal mathematical settings. Diagrams can, for instance, reify formal mathematical concepts to make them meaningful for students (Alcock & Simpson, 2004; Gibson, 1998), allow students to quickly draw useful inferences (Alcock & Simpson, 2004; Gibson, 1998), and help students overcome the impasses they reach when writing a proof (Gibson, 1998; Weber & Alcock, 2004). Proofs whose construction is based on diagrams may be more meaningful for students (Raman, 2003) and provide them with more learning opportunities (Weber, 2005) than proofs based on formal deduction alone. Empirically, there are case studies documenting undergraduates' success in producing proofs based on diagrams (Alcock & Weber, 2010a; Gibson, 1998), demonstrating that a diagrammatic approach to writing proofs can be useful for some students in some situations. Furthermore, there is ample empirical evidence that mathematicians frequently base their proofs on diagrams (e.g., Burton, 2004; Hadamard, 1945; Thurston, 1994). If students could engage in similar processes when they write proofs, their proving and reasoning processes might resemble more closely those used by mathematicians.

**1. 4. Limitations to the effectiveness of basing proofs on diagrams**

Although there is good reason to believe that undergraduates should base some proofs on diagrams, there are several factors that may limit the effectiveness of this pedagogical suggestion.

Many researchers have delineated theoretical difficulties that might prevent students from successfully basing a proof on a diagram. For instance, Duval (2007) notes that the structure of an visual argument (or any informal argument) differs significantly from the structure of a formal proof; he cautions researchers not to underestimate the cognitive complexity of the task of determining the statuses of assertions within a proof (i.e., is the assertion an assumption, an axiom, an accepted fact, or a deduction?) and organizing the chain of deductions appropriately. This can be particularly difficult when basing a proof on a diagram as each assumption may appear equally obvious, making it hard for students to distinguish what can be assumed and what needs to be shown (cf. Alcock & Simpson, 2004).

There are theoretical difficulties beyond cognitive complexity with using diagrams effectively in proof construction. Alcock and Simpson (2004) note that diagrams sometimes provide students with unwarranted confidence in the conjectures that they make, leading them to believe things that are not true or feeling that a proof of a true statement is superfluous. Alcock and Weber (2010a, 2010b) argue that students often have inaccurate representations of mathematical concepts or fail to see connections between their visual representations of a mathematical concept and the concept's formal definition, making it difficult or impossible to base formal proofs of those diagrams.

Moreover, although some published case studies in mathematics education illustrate how some undergraduates are able to use insight gleaned from diagrams to construct proofs, other case studies illustrate how undergraduates often are not able to do so. For instance, Alcock and Weber (2010b) asked eleven mathematics majors to prove that an increasing function does not have a global maximum. The four students who drew diagrams of a generic increasing function each instantly gained conviction that the statement to be proven was true but none of these students could not see how to begin writing a proof of this claim. Finally, several small-scale studies have found little to no correlation between undergraduates' propensity to use diagrams and their success in proof-writing in advanced mathematics (e.g., Alcock & Simpson, 2004, 2005). These findings call to question the claim that the unqualified suggestion to increase diagram usage will improve students' abilities to construct proofs. If we would like the benefits of diagrams that mathematicians exploit to be accessed by students, then further work must be done to determine the nature of how and when mathematicians use diagrams.

**1. 5. Research questions**

We find promise in the suggestion that students learn to use diagrams. However, we contend that students will need guidance in order to implement this suggestion effectively. Reflecting on her own teaching, Alcock (2010) writes:

> "Diagrams can provide insight, but it is not always easy for students to make detailed links between what is in the diagram and what is in a formal proof. This means that the step between seeing that a result must be true and proving it can seem insurmountable. Through my small-class teaching, I have also learned that students often find it difficult to draw a diagram based on verbal and algebraic information. As a result, I now spend more time walking students through the process of drawing diagrams" (p. 232-233).

We share Alcock's goal of trying to make explicit the process that connects the construction of formal proofs and the gaining of conviction from diagrams. The study reported in this paper contributes to this goal by examining how eight mathematicians use diagrams in their own proof-writing. Three questions drove our analysis:

1. To what extent did these mathematicians base their proofs on the diagrams?
2. How, and for what purposes, did these mathematicians use diagrams to assist their proof writing?

## 2. Methods

### 2. 1. Participants

Eight mathematicians participated in this study. All mathematicians currently work either in post-doctoral positions or in tenure-track positions at a research institution in universities with strong research mathematics programs in the United States. The participants' areas of research included algebra, analysis, geometric topology, and combinatorics. The range of experience of the participants also varied, ranging from 1 year to 30 years. In this report, we use pseudonym initials to refer to the mathematicians and refer to all participants with the masculine pronoun to further protect their identities.

### 2. 2. Materials

Participants were asked to prove a claim about the sine function:

*Show that restrictions to the sine function of intervals of length greater than π cannot be injective.*

This task, which we refer to as the Sine Task, features the following characteristics. First the claim is based on relatively elementary mathematics—the sine function and injectivity are part of most secondary mathematics curricula. Thus the meaning of this claim would be accessible to sophomore and junior undergraduate mathematics majors. Second, proving this claim invites the use of a diagram. We believed that upon reading the claim, most participants would sketch the graph of a sine function. In Section 3.1, we report that seven of the eight participants did construct this graph shortly after reading the claim. Third, we believed that it would be fairly easy to accept the claim as true after viewing the graph of a sine function, based on pilot data that was collected. Hence the participants would not need to spend a long time verifying the claim is correct; rather most of the work in proving this claim would involve creating an argument that established the claim. As we will show in Section 3.1, six participants indicated that they were convinced that the claim was true early on in the interview session. Fourth, despite the relative simplicity of the mathematical concepts involved and the apparent truth of the claim, we believed that writing a full proof of the claim would not be trivial. We will show in Section 3.2 that many of the participants in this study did indeed have some difficulty constructing a complete proof.

We believe that, as a consequence of the above characteristics, observing mathematicians' performance on this task has the potential to shed insight on the processes that mathematicians use to construct a proof based on a diagram in a mathematical domain accessible to undergraduate mathematics majors.

### 2. 3. Procedure

Participants met individually with the second author for a one-hour interview. All interviews were videotaped. Participants were handed the task described above. Participants were told to write a proof of this claim appropriate for an undergraduate textbook for mathematics majors. After producing a proof, participants were invited by the interviewer to revise the proof. Participants were not given any time limit on the task, and were allowed the opportunity to revise their final proofs if they thought it necessary.

**2. 4. Analysis**

The participating mathematicians collectively made a total of 321 mathematical statements. The coding of this data was conducted in the style advocated by Weber and Mejia-Ramos (2009) to analyze the contributions that different types of reasoning played in the construction of a proof. A statement was coded as an *assertion* if it was a mathematical statement that could be validated as true or false (either in an absolute sense, or within the context of previous assumptions made by mathematicians). An example of an assertion is "$\sin(x + \pi) = -\sin x$". A statement was coded as a *proof approach* if it suggested a way to proceed with the proof considered by the participant, for example, a suggestion to use a proof by cases or to employ techniques from calculus. A statement was coded as an *evaluation* if the participant judged the validity of a previously stated assertion (e.g., "$\sin(x + \pi) = \sin x$ is false"), the utility of a previously stated assertion, or the viability of a previously stated proof approach. We were able to code each mathematical statement into one of these three categories.

We next considered the source of each of the statements that participants made. We attributed statements to three sources. A *deduction* was a statement made by the participant that was a logical consequence of previous statements. Deductions included instances when a participant specifically referred to previous statements that he or she had made, indicated they were making a new deduction by using a connector such as "hence" or "consequently", or uttered a statement that was an immediate mathematical consequence of the statement that preceded it. A statement was coded as an *inference from a diagram* if the participant cited, either verbally or with gesture, a graph of the sine function or another diagram that he or she produced as a basis for making that assertion. A statement was coded as *recall or unknown* if the participant neither referenced a graph or diagram that he or she constructed nor cited previous assertions as a basis for making the new assertion.

The participants collectively drew 32 inferences from the diagrams that they constructed. To classify the types of inferences that the participants drew, we used an open coding scheme in the style of Strauss and Corbin (1990). We first wrote a short description of each inference. Similar inferences were grouped together and given preliminary category names and definitions. New inferences were placed into existing categories when appropriate, but also used to create new categories or modify the names or definitions of existing categories. This process continued until a set of categories was formed that were grounded to fit the available data. Descriptions and illustrations of each category will be provided in Section 3.3.

Finally, we coded the correctness of the participants' final proofs using the coding scheme of Malone et al (1980) that was used by Hart (1994) and Weber (2006) in their studies evaluating the correctness of students' proofs. Each proof was coded as completely correct, correct except for minor and insignificant errors, an incorrect proof with significant errors but substantial progress, or an incorrect proof with significant errors where no significant progress was made. Typed versions of two examples of proofs that were coded as completely correct are presented in the Appendix.

For each participant, the result of this analysis was a table containing every mathematical statement that the participant made and the type of statement that was made (inference from a diagram, deduction, or recall/unknown). For inferences from diagrams, the purpose and nature of this inference was described. For deductions, we could highlight the previous statements on which that deduction was based. These tables allowed us to infer how, and to what extent, diagrams played a role in the participants' proof construction process.

## 3. Results

A summary of the participants' behavior is presented in Table 1.

| Mathematician | Drew a diagram? | Quality of proof | Time on task (minutes) | # of inferences from the diagram | Types of inferences made |
|---|---|---|---|---|---|
| TL | Y | Mostly correct | 30 | 10 | NP, RI, SPA |
| CY | Y | Mostly correct | 11 | 2 | NP |
| PV | Y | Correct | 9 | 7 | NP, ET |
| FR | Y | Incorrect with substantial progress | 18 | 2 | NP, ET |
| IR | Y | Correct | 11 | 3 | NP, ET |
| BT | Y | Needed substantial hints | 21 | 2 | RI, SPA |
| KZ | Y | Correct | 33 | 6 | NP, RI, ET, SPA |
| KT | N | Correct | 14 | 0 | |

**Table 1.** A summary of mathematicians' proofs. (Key: NP=Noticing Properties/Generating Conjectures, RI=Representing/Instantiating Ideas or Assertions, ET=Estimating the Truth of an Assertion, SPA=Suggesting a Proof Approach)

### 3. 1. The extent of participants' diagram usage

We predicted that the Sine Task would invite the use of a diagram and this diagram would provide participants with confidence that claim in this task was correct. The participants' behavior verified these assumptions. Seven of the eight participants drew a graph of the sine function shortly after reading the problem statement. Six participants indicated they were convinced that the claim was true either prior to or immediately after drawing the graph.

Three participants drew diagrams aside from the graph of the sine function. KZ drew a right triangle while attempting to verify that $\sin(\theta) = \cos(\pi/2 - \theta)$; later, he drew a number line to help him justify that every interval [a, b] of length strictly greater than $\pi$ contained a point of the form $k\pi/2$ for some odd integer k. TL drew a portion of the graph of an arbitrary function containing a local maximum to prove functions cannot be injective on intervals containing local maxima. When BT reached an impasse in writing the proof and did not know how to proceed, he drew a unit circle and made two inferences from this diagram.

The diagrams played a substantial role in four of the participants' proof constructions. TL and KZ repeatedly drew inferences from the diagrams, both to suggest the proof techniques they ultimately based their proofs upon and to recognize properties of the sine function that were used in their proofs. IR and PV used their diagrams to draw inferences that were used in their final proofs, as well as to gain conviction in these deductive inferences.

In contrast, three participants did not make extensive use of their diagrams in their proof constructions. KT did not draw a single inference from a diagram. FR used his diagram solely to identify the period of the sine function, an idea that did not play a pivotal role in his final proof. As mentioned above, BT did draw two inferences from his diagram, but these were ultimately not connected to his final proof.

Finally, CY exhibited behavior between these two extremes, using the diagram to notice relevant properties of the sine function. Differential usage of diagrams in proof construction tasks has been reported in other small-scale studies, such as Raman's (2003) study on how mathematicians used graphs to prove that the derivative of an even function was an odd function.

**3. 2. Participants' difficulties with the Sine Task**

We predicted that constructing the required proof would not be trivial. The data displayed in Table 1 suggests this expectation was correct.

The task was not straightforward for the participants as a whole. Participants spent a mean time of 18 minutes working on this task with two participants spending more than 30 minutes on the task. While working on the proof construction, seven of the eight participants made assertions that were either false or assertions that were true but irrelevant to the proof that they produced. Only KT's proof construction proceeded in a direct and linear fashion.

Even given our predictions about task complexity, we were surprised by the extent to which the task posed more difficulties for these participants than we anticipated. It is of course hard to perform mathematics, while thinking aloud, and make only absolutely correct statements. Yet despite an invitation to revise the proof, and despite the length of time spent on the task, only four of the eight participants produced a completely correct proof. Two participants had what we judged to be minor errors in their proofs. TL argued that the arbitrary interval of length greater than $\pi$ could be translated back using the periodicity of the sine function to contain the point $x = \pi/2$. However, this is not necessarily true. For instance, the interval $(-\pi, \pi/4)$ has length greater than $\pi$ but cannot be shifted by a multiple of $2\pi$, the period of sin(x), to contain the point $x = \pi/2$. (TL's argument could be salvaged by noting that the interval can be translated so it either contains the point $x = \pi/2$ or $x = -\pi/2$.) CY noted correctly that any closed interval of length greater than $\pi$ would have to contain a local maximum or minimum for the sine function, but neglected to specify that one could find a maximum or minimum in the interior of this interval, a fact that would be necessary for his subsequent argument to work.

BT and FR had more serious difficulties with constructing their proofs. BT reached an impasse after trying unsuccessfully for over 11 minutes to construct a proof. At this point, he asked for hints on how to proceed and used substantial hints to produce the proof. FR produced the proof in which he uses Rolle's Theorem to deduce that sin(x) cannot be injective on an interval where its derivative is zero. However, Rolle's Theorem asserts that given real numbers a, b with $a<b$, if a continuous differentiable function f satisfies $f(a) = f(b)$, then there exists a real number c such that $a<c<b$ and $f'(c) = 0$. FR was actually applying the false converse of Rolle's Theorem, assuming that if there were a c such that $f'(c) = 0$, and f were defined on an interval around c, there must be distinct points a and b such that $f(a) = f(b)$. (Note for instance, this claim is false for $f(x) = x^3$, although it is true in the case that FR considered)

There were also cases in which participants had the essential idea for how their proof would proceed yet still had difficulty in finalizing the proof. One case of this is TL's proof construction. After working for some time, TL verbally articulated a rough sketch of a proof of the claim. In this sketch, he claimed that a continuous function cannot be injective on a neighborhood of a local maximum, which "follows immediately from the definition of what it means to be a local maximum." Yet, when TL began to write down the proof of the original claim, he stopped to pursue a proof of this sub-claim, drawing the top diagram in Figure 1, and uttered the following:

TL: Let me think for a minute. [draws middle diagram in Figure 1] OK, so let me see. So this is going to be something like Rolle's Theorem or the Intermediate Value Theorem...is basically the tool that I have to use, but I just have to figure out the right way of saying it. I don't think I can just appeal to just a theorem to do it...

In this passage, TL drew diagrams of what appeared to be generic continuous functions, and although he seems confident of the main tool that will prove this sub-claim, appears to be unsure of exactly how to proceed.

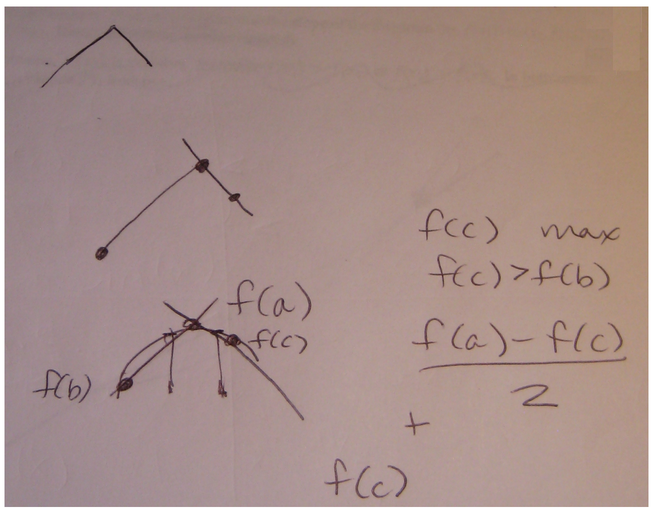


**Figure 1.** TL's generic continuous functions with local maxima.

After reasoning with these diagrams for nearly four minutes, TL eventually develops an argument for this sub-claim by picking points, *b,* and *c*, to the left and right, respectively, of *a,* the location of the local maximum, and applying the Intermediate Value Theorem to show that the function fails to be injective. However, TL's argument could have been more straightforward —instead of simply using the maximum of f(b) and f(c) as the value which is mapped onto twice in the interval, TL constructs a quantity which is half the distance from f(a) to max{f(b), f(c)} which he then uses to obtain the value which is mapped onto twice.

The above transcript, the time spent on developing this sub-argument, and the extraneous construction in the final solution all indicate that although TL had suggestive diagrams and an informal argument of the original claim, substantial work remained to establish the details required for a formal proof.

**3. 3. Purposes of diagrams in participants' proof constructions**

The purposes of participants' diagrams are summarized in Table 2.

| Purpose served by diagram | Total # of instances | Participants who used the diagram for this purpose |
|---|---|---|

| | | |
|---|---|---|
| Noticing properties and generating conjectures (NP) | 11 | TL, CY, PV, FR, IR, KZ |
| Representing/Instantiating an idea or assertion (RI) | 6 | TL, KZ, BT |
| Estimating the truth of an assertion (ET) | 9 | PV, FR, IR, KZ |
| Suggesting a proof approach (SPA) | 6 | TL, KZ, BT |

**Table 2.** A summary of mathematicians' purposes of using diagrams.

### 3. 3. 1. Noticing properties and generating conjectures

There were 11 instances when a participant used a diagram to notice a mathematical property or make a conjecture that might plausibly be useful for proving the claim. In some cases, the participant conjectured that a property that might be true but would have to be proven, as is illustrated in the excerpt below:

CY: I'm sort of drawing the graph. I'm trying to figure out is there some property which two points which differ by π have. Can you write down some expression for that in terms of the sine curve? So I drew a picture of the curve. … Let's see. It looks a bit like they're negative of one another. Is that true? Do I know how to prove that? I should write down something. Does sin(x) equal –sin(x+π)?

In the excerpt above, CY notices that sin(x) appears to be equal to –sin(x + π) but is not certain that this is correct and begins thinking about how to prove that. In other cases, the participant gained a high level of conviction that the property they noticed was correct after inferring it from the diagram.

TL: [while gesturing to graph in Figure 2]. OK, so clearly, let's see. If the interval strictly contains this point, π/2 or minus π/2, then you can see from these areas... So we take this interval and we translate it back, so if that interval contains either one of these, then it's not going to be injective, sort of obviously.

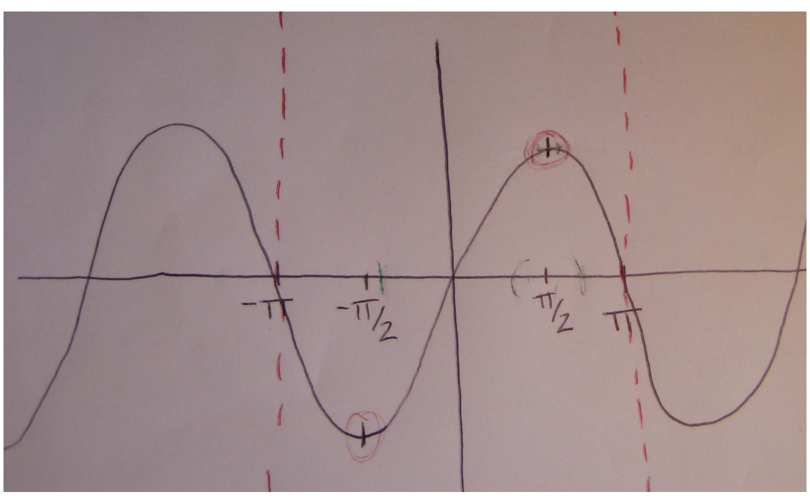


**Figure 2.** TL's graph of the sine function.



In this excerpt, TL uses what he drew in Figure 2 to see that the interval [a, b] could be translated to contain an interval with $\pi/2$ or $-\pi/2$ and the sine function would "sort of obviously" not be injective on this interval.

### 3. 3. 2. Estimating the truth of an assertion

In 9 instances, participants referred to a diagram to see whether a statement they made was correct. In some cases, these statements were conjectures; by inspecting the diagram, the participant could either gain conviction to accept the conjecture as true or find sufficient reason to reject the conjecture as false. We illustrate the latter with the excerpt below.

> PV: The sine function looks like this [draws a graph of the sine function] and this length is $\pi$ [draws length] … Yeah, so if you have an interval of length greater than $\pi$, you can see the top and you can see the bottom…That's not true…

In the excerpt above, PV initially asserts that any interval of length greater than $\pi$ would contain both a local maximum and a minimum for the sine function (or, in his words, "you can see the top and you can see the bottom"), but upon inspecting the diagram further, he realizes that this is not true. In a similar instance, FR initially asserted that $\sin x = \sin(x + \pi)$, but after drawing a graph quickly realized the period of sine was actually $2\pi$ and thus rejected this assertion.

In other cases, participants would look at the graph to verify inferences that they had reached previously via deductive reasoning in case they made an error at some point. In the strongest illustration of this, KZ initially recalled that $\sin \theta = \cos(\pi/2 - \theta)$ but was not sure if this formula was correct. To verify that this was correct, he said, "so this can be seen if you think of sine from the triangle" and drew a right triangle with one acute angle labeled $\theta$ and the other, $\pi/2 - \theta$. After drawing this triangle, KZ averred, "so this should be true in general". However, when he tried to prove the claim, he misremembered the cosine addition and subtraction identities and incorrectly deduced that $\cos(\pi/2 - \theta) = -\sin(\theta)$. This deduction troubled him and he was not certain whether $\cos(\pi/2 - \theta)$ was equal to $\sin \theta$ or $-\sin \theta$. To decide which was correct, KZ sketched a graph of the sine and cosine functions and found support for his original claim. Then, returning to his incorrect identity, he checked specific values of $\theta$. Only after realizing that the identity did not hold for some values did he reject the incorrect identity and decide to retain the original claim.

This illustrates an interesting aspect of how mathematicians seek and gain conviction. It is sometimes claimed in the mathematics education literature that while empirical and visual evidence can provide support for mathematical proofs, it is deductive reasoning that provides complete certainty in a claim. For instance, Harel and Sowder (1998) state that mathematicians have deductive proof schemes, meaning that they obtain complete conviction from the deductive proofs that they produce. This is in contrast to students who hold empirical or perceptual proof schemes and believe claims are true via the inspection of diagrams or examples. As KZ's behavior illustrates, the situation is not always so straightforward. He initially gains confidence that $\cos(\pi/2 - \theta)$ equals $\sin \theta$ using the right triangle he drew. However, he doubts the veracity of this assertion after deducing a contradictory identity. What is interesting is that his (invalid) deduction did not have the last word. Rather he sought other diagrammatic evidence to resolve this contradiction and only obtained full conviction after checking his original identity with specific numbers. This suggests that while the goal of mathematicians is often to produce rigorous proofs of mathematical theorems, they might obtain conviction in these theorems via a combination of empirical, diagrammatic, and deductive reasoning; and deductive reasoning is not always the sole or dominant way of resolving mathematical questions, at least not in short-term episodes in their problem-solving work.

### 3. 3. 3. Suggesting proof approaches

In six instances, KZ and TL used the diagram to suggest ways that their proofs might proceed. We illustrate this with an episode of TL's work:

> TL: So I already have the picture of it [referring to Figure 2, presented earlier in this section], well let's see, so it's a periodic function. So, whatever interval that you were looking at – if you take some interval of length strictly greater than π, you can always translate it backwards so that it's around the origin, because sine is periodic. OK, so here's a period of the sine function from here [draws the two dashed lines on the sine graph in Figure 2]. OK, and we want to know about intervals of length strictly greater than π. So if we take any interval at all, I can move it back to this region, and it's necessarily going to contain the origin because length of the interval is strictly greater than π. I guess this is an easier proof than taking derivatives, but maybe it's not quite a proof yet. I have to think about it a little more carefully. OK. So we take any interval of length strictly greater than π and move it back into this fundamental region here ... So the question is I have to make a precise way of saying what it is that I'm talking about, but I can see how the argument is forming.

In this excerpt, TL lays out how his proof will proceed. He will use the periodicity of the sine function (a property he notices from his inspection of the graph of sine) to map the interval he is considering into the region indicated in Figure 2.

### 3. 3. 4. Instantiating or representing an idea or assertion on the graph

There were six instances in which TL or KZ represented an idea or assertion graphically. Consider the excerpt below:

> KZ: So this interval [a,b] must contain an element of the form kπ/2 because [begins drawing a number line] we're talking about π/2, 3π/2, 5π/2 and so on, the distance between those is π [labels these points on the number line as he states them]. So if you take any interval of length strictly bigger than π, it must contain at least one of them. [draws the interval of length greater than π on the number line] And moreover, it must contain some interval close to them too, both on the left and on the right, and you can choose a θ close enough there.

Throughout this excerpt, KZ is representing the ideas that he says aloud on his diagram of the number line. In particular, at the end of this excerpt, KZ chooses and draws an interval of length strictly greater than π. We believe one purpose of instantiating these ideas is to aid in the generation of new conjectures and potential proving approaches.

## 4. Discussion

The fact that we only looked at eight mathematicians proving a single claim limits the extent that we can generalize our findings. Nonetheless the data do reveal several interesting themes about mathematicians' use of diagrams in proof-writing.

The first theme we observed is that the participants had surprising amount of difficulty in producing complete proofs of the claim. As discussed earlier in this paper, difficulties occurred even in cases in which participants had constructed what might be called the crux or key idea of the proof. In Section 3.1, we describe how TL struggled to write a proof of a claim in his argument even after both establishing an informal argument of the original claim and drawing suggestive diagrams of the situation. In Section 3.2, we described how KZ had significant difficulties completing a symmetry-based argument because of trouble he had justifying an important trigonometric identity.

That these mathematicians did not find it trivial to translate an informal argument to a formal one illustrates an important point. Researchers such as Hanna (1991) have claimed that the essence of a proof is contained in the intuitive ideas used to try to develop the formal proof. While Hanna acknowledges that having a logically correct formal argument is important, she refers to the full rigor as a "hygiene factor", implying that translating the intuitive idea to a formal proof is a relatively uninteresting and routine part of the proof writing process for mathematicians. Although this is probably true, the data from this study remind us that this translation still introduces difficulties. If mathematicians experience these difficulties, it stands to reason that students will as well. Hence this study serves as a reminder that, although the generation of informal visual arguments to serve as a basis for formal proofs is a laudable goal in the advanced mathematics classroom, care still must be taken in describing the complex process of translating these arguments into a proof (e.g., Alcock, 2010; Duval, 2007).

The participants used diagrams for four purposes: (a) noticing properties and generating conjectures, (b) suggesting a proof approach, (c) estimating the truth of an assertion, and (d) instantiating an idea or assertion. In our previous studies on students' proof constructions using diagrams, we noticed that participants used diagrams primarily to try to understand a mathematical claim and sometimes to generate an explanation of why a theorem is true (see Alcock & Weber, 2010a, 2010b). However these students generally did not use diagrams for some of the reasons that the mathematicians did, such as verifying that their logical deductions were true, uncovering errors in their logic, or identifying false mathematical assertions. Further, in mathematics lectures that we have observed, these aspects of proof construction were often absent (see, for instance, Weber, 2004). For instance, the professors we have observed rarely show the wrong turns that they make in a proof attempt or how a diagram can reveal an error in their logical reasoning. These observations suggest a gap between mathematicians' use of diagrams in their work, and their use of diagrams in teaching. If the goal of the pedagogical suggestion of basing proofs on diagrams is to help students benefit from diagrams similarly to how mathematicians benefit, then we must find ways to raise students' awareness of and access to the variety of potential uses of diagrams in proof construction.

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**Appendix.**

In what follows, we give two correct proofs from our participants PV and KT. Both proofs were lightly edited for the purposes of readability and brevity but the central ideas employed in the proofs were unchanged.

Proof 1—symmetries of the sine function:

Every interval of length greater than $\pi$ contains a point, $x_0$, of the form $\pi/2 + \pi k$ for some integer k, and a neighborhood, I, containing $x_0$ . Now, sine is symmetric about $\pi/2 + \pi k$ in the sense that $\sin(\pi/2 + \pi n + x) = \sin(\pi/2 + \pi n - x)$ for all real x and integral n, as can be checked using the sine angle addition formula. Choose $\varepsilon$ such that $x_0 - \varepsilon$ and $x_0 + \varepsilon$ are elements of I. Then $\sin(x_0 + \varepsilon) = \sin(\pi/2 + \pi k + \varepsilon) = \sin(\pi/2 + \pi k - \varepsilon) = \sin(x_0 - \varepsilon)$, so sine is not injective on I, and hence not injective on the original interval.

Proof 2—Intermediate Value Theorem:

Every interval of length greater than $\pi$ contains a point, $x_0$ of the form $\pi/2 + k\pi$ for some integer k, and a neighborhood, I, containing $x_0$. If $f(x) = \sin(x)$, $f'(x) = \cos x$ and $f''(x) = -\sin x$. Note that $f'(x_o) = 0$ and $f''(x_o) \neq 0$. Hence f attains a relative maximum or minimum at $x_0$. Assume f attains a relative maximum at $x_0$ (a similar argument works if $x_0$ is a relative minimum). Let a, b be in I such that $a < x_0 < b$. Let $y_0 = \max(f(a), f(b))$. Then, since $y_0$ is a relative maximum $y_0 <= f(x_0)$. Also, $f(a) <= y_0$ and $f(b) <= y_0$ by construction. Since f is continuous, we may apply the Intermediate Value Theorem: there exists a' in $[a, x_0]$ and b' in $[x_0, b]$ such that $f(a') = y_0 = f(b')$. Hence, $f(x) = \sin(x)$ is not injective on I, and hence not injective on the original interval.